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Partially and fully frustrated coupled oscillators with random pinning fields

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Abstract

We have studied two specific models of frustrated and disordered coupled Kuramoto oscillators, all driven with the same natural frequency, in the presence of random external pinning fields. Our models are structurally similar, but differ in their degree of bond frustration and in their finite size ground state properties (one has random ferro- and anti-ferromagnetic interactions and the other has random chiral interactions). We have calculated the equilibrium properties of both models in the thermodynamic limit using the replica method, with emphasis on the role played by symmetries of the pinning field distribution, leading to explicit predictions for observables, transitions and phase diagrams. For absent pinning fields our two models are found to behave identically, but pinning fields (provided with appropriate statistical properties) break this symmetry. Simulation data lend satisfactory support to our theoretical predictions.

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1. Introduction

Models of interacting oscillators have received much attention during the last few decades [1]. They have been studied in many fields, including physics, chemistry and biology [2]. One of the most successful models was proposed by Kuramoto [3], who assumed that each oscillator moves in a globally attracting limit cycle of constant amplitude, with interactions between the oscillators which are sufficiently weak to ensure that perturbations will not move them away from their individual limit cycles. Only one degree of freedom, the phase θ_i, is then required to describe the dynamics of each oscillator. The phases evolve according to $\frac{\mathrm{d}}{\mathrm{d}t}\theta_i = \omega_i + \sum_j K_{ij} f(\theta_i - \theta_j)$. Here ω_i is the natural frequency of oscillator i, K_{ij} is the coupling strength between oscillators i and j and $f(x)$ is a nonlinear function, Kuramoto initially considered mean field coupling strengths $K_{ij} = K/N$, where N is the total number of oscillators, and $f(x) = \sin(x)$. This particular model has been investigated intensively. Generally, for some critical (positive) value of K a phase transition occurs from a state where all the oscillators run incoherently to a state where a degree of synchronization emerges spontaneously. The details of this transition depend on the the distribution of natural frequencies [4–7] and the nature of the interactions (namely, short range interactions [8, 9], random site disorder [10] or even more complex types [11]). Kuramoto-type models have also been studied extensively in the presence of noise [4, 12, 13].

In the context of neural networks [14, 15] and Josephson junctions [16–18] one also encounters Kuramoto's equations, but usually with an extra term A_{ij} in the argument of the nonlinear function: $\frac{\mathrm{d}}{\mathrm{d}t}\theta_i = \omega_i + \sum_j K_{ij}\sin(\theta_i - \theta_j + A_{ij})$. The simplest case, $A_{ij} = \alpha$ for all (i, j), was studied in [19]. Here the degree of spontaneous synchronization was found to be maximal when $\alpha = 0$, decreasing monotonically to zero as $\alpha \to \frac{\pi}{2}$. A more complicated case is the subject of [20]. However, the problem is far from solved for the case of quenched bond disorder. Here only simulation studies [21, 22] have been published; these emphasize the crucial role played by the disorder in determining long time properties of the model, as measured by correlation functions.

The simplest models with random interactions and/or pinning fields are those where all oscillators have the same natural frequency ω_i; here the latter can be transformed away. If the matrix $\{A_{ij}\}$ fulfils certain symmetry criteria, then Kuramoto's equations describe a gradient descent process, enabling (upon adding Gaussian noise) an equilibrium statistical mechanical analysis. Low dimensional versions of this system are equivalent to frustrated XY models, used to describe Josephson junction arrays. Here the sum over interacting pairs in the Hamiltonian is restricted to neighbours which form a plaquette, with θ_i denoting the spin orientation at site i, and A_{ij} denoting the bond angle, such that $\sum_{(i,j)\in\mathrm{plaq}} A_{ij} = 2\pi f$. For $f = 1/2$ one has the so-called fully frustrated model. The critical behaviour and the symmetries of these low dimensional systems depend highly discontinuously on f [23]. Adding random fields gives rise to a complex phenomenology which is still the subject of study. It would be interesting to investigate whether such features remain in mean-field models. The effect of random fields has so far been analysed mostly in systems without quenched disorder [24], or for m-vector spin glass models [25–27] (but for local uniaxial anisotropic fields of a different nature than those discussed here). In [28] one finds a mean-field model with disordered bonds and pinning fields, but with simpler site permutation symmetries and disorder distributions, and more limited analysis than in the present paper.

Our paper is structured as follows. We first define two mean-field models with disorder in bonds and pinning fields, and their order parameters. The first is of a conventional type, with pairs of oscillators trying to either synchronize or anti-synchronize, depending on the sign of their interaction. The second model is less conventional; here oscillator pairs prefer phase differences of either $\pi/2$ or $-\pi/2$. For both models the presence of random pinning fields increases energetic conflicts and frustration. In section 3 we solve both models in equilibrium, using the replica method; we make the replica-symmetric ansatz and calculate the conditions for replicon instabilities. In sections 4 and 5 we study the effects of global symmetries. In section 6 we present results (phase diagrams and the temperature and field strength dependence of observables) for a number of specific choices for the pinning field distributions, and validate our predictions via numerical simulations.

2. Model definitions

We study systems of N coupled Kuramoto oscillators (or XY spins) with external pinning fields, described by Hamiltonians of the form

$$H = -\frac{2K}{\sqrt{N}} \sum_{i<j} \cos(\theta_i - \theta_j + A_{ij}) - h \sum_i \cos(\theta_i - \phi_i) \tag{1}$$

where $\theta_i \in [0, 2\pi]$ denotes the phase of the ith oscillator. We introduce disorder by drawing the pinning angles ϕ_i independently at random from a distribution $p(\phi)$, and the relative angles A_{ij} independently at random from a distribution $P(A)$, with $\int \mathrm{d}A\ P(A)\cos(A) = 0$ (to ensure that the bond disorder will retain significance for $N \to \infty$). This system will generally exhibit competition between alignment to the pinning fields and the realization of prescribed relative angles between pairs. We simplify the bond average by choosing binary $\{A_{ij}\}$: $P(A) = \frac{1}{2}\delta[A - A^*] + \frac{1}{2}\delta[A - A^* - \pi]$. Without loss of generality we may take $-\frac{\pi}{2} < A^* \leqslant \frac{\pi}{2}$. However, only for $A^* \in \left\{0, \frac{1}{2}\pi\right\}$ will it be possible to calculate the disorder-averaged free energy for the system (1) in terms of a standard replica mean field theory; these two cases are the subject of our paper:

$$\text{model I:} \quad H = -\frac{2K}{\sqrt{N}} \sum_{i<j} J_{ij} \cos(\theta_i - \theta_j) - h \sum_i \cos(\theta_i - \phi_i) \tag{2}$$

$$\text{model II:} \quad H = -\frac{2K}{\sqrt{N}} \sum_{i<j} J_{ij} \sin(\theta_i - \theta_j) - h \sum_i \cos(\theta_i - \phi_i) \tag{3}$$

with $J_{ij} \in \{-1, 1\}$. In (2) energy minimization will translate for any oscillator pair (i, j) into the objectives

$$J_{ij} = 1: \quad \theta_i - \theta_j \to 0 \qquad (\text{mod } 2\pi)$$
$$J_{ij} = -1: \quad \theta_i - \theta_j \to \pi \qquad (\text{mod } 2\pi)$$

(describing ferro- and anti-ferromagnetic interactions). A triplet (i, j, k) is bond-frustrated if and only if an odd number of the relative angles involved equals -1. In (3) energy minimization will translate for any pair (i, j) into the objectives

$$J_{ij} = 1: \quad \theta_i - \theta_j \to \pi/2 \qquad (\text{mod } 2\pi)$$
$$J_{ij} = -1: \quad \theta_i - \theta_j \to \pi/2 \qquad (\text{mod } 2\pi).$$

Now *every* triplet (i, j, k) is bond-frustrated.

It is not *a priori* clear whether and how the solutions of models (2) and (3) should be related. For instance, finding the ground states for $N = 3$ and $h = 0$ reduces to minimizing $H_{\rm I}(\theta_1, \theta_2) = J_1\cos(\theta_1) + J_2\cos(\theta_2) + J_3\cos(\theta_1 - \theta_2)$ and $H_{\rm II}(\theta_1, \theta_2) = J_1\sin(\theta_1) + J_2\sin(\theta_2) + J_3\sin(\theta_1 - \theta_2)$, respectively (rotation invariance allows us to put $\theta_3 = 0$), with $J_i \in \{-1, 1\}$ (chosen randomly). Solving this simple problem reveals that the ground state energy $E_{\rm I}$ of model I is dependent on the bonds $\{J_i\}$, $E_{\rm I} \in \{-1, -3\}$. Averaging over the bond realizations gives $\overline{E_{\rm I}} = -2$. In contrast, for the ground state energy of model II one finds $E_{\rm II} = -\frac{3}{2}\sqrt{3}$, independent of the bond realization.

Pinning fields break the global rotation invariance of (1) which is present for $h = 0$. Any nonzero value of the average pinning direction $\langle\phi\rangle_\phi$ can be transformed away, via $\theta_i \to \theta_i + \langle\phi\rangle_\phi$, so without loss of generality we may choose $\langle\phi\rangle_\phi = 0$. In this paper we restrict ourselves, for simplicity, to reflection-symmetric pinning field distributions, where

$p(\phi) = p(-\phi)$ for all $\phi \in [-\pi, \pi]$. However, the free energies of our models ((2) and (3)) are both invariant[3] under the transformation

$$p(\phi) \to \frac{1}{2}(1+\eta)p\left(\phi - \lambda - \frac{\pi}{2}\right) + \frac{1}{2}(1-\eta)p\left(\phi - \lambda + \frac{\pi}{2}\right) \tag{4}$$

for any $\eta \in [-1, 1]$ and $\lambda \in [0, 2\pi]$, so we solve implicitly for a larger family of models. For high temperatures we expect the symmetries of $p(\phi)$ to be inherited by the solutions of our models. We define a complex (single-site) disorder-averaged susceptibility χ, upon adding small perturbations to the external pinning angles ϕ_i:

$$\chi = \frac{1}{\mathrm{i}Nh}\sum_i \overline{\left[\mathrm{e}^{-\mathrm{i}\phi_i}\frac{\partial}{\partial\phi_i}\langle \mathrm{e}^{\mathrm{i}\theta_i}\rangle\right]}. \tag{5}$$

Here $\langle\cdots\rangle$ and $\overline{[\cdots]}$ denote equilibrium and disorder averages, respectively. For perfectly linear single-site response, i.e. $\langle \mathrm{e}^{\mathrm{i}\theta_i}\rangle = x\,\mathrm{e}^{\mathrm{i}\phi_i}$, definition (5) would give $\chi = x/h$. Evaluating (5) for systems with Hamiltonians of the form (1) gives

$$\chi = \frac{\beta}{\mathrm{i}N}\sum_i \overline{[\langle \sin(\theta_i - \phi_i)\,\mathrm{e}^{\mathrm{i}(\theta_i-\phi_i)}\rangle - \langle \sin(\theta_i-\phi_i)\rangle\langle \mathrm{e}^{\mathrm{i}(\theta_i-\phi_i)}\rangle]}. \tag{6}$$

3. Equilibrium analysis

3.1. Solution via replica theory

We calculate the disorder-averaged free energy per spin $\overline{f} = -\lim_{N\to\infty}(\beta N)^{-1}\overline{\log Z}$ for the Hamiltonian (1) with $P(A) = \frac{1}{2}\delta[A - A^*] + \frac{1}{2}\delta[A - A^* - \pi]$, using the standard procedures of conventional replica theory [30]. This gives

$$\overline{f} = -\lim_{N\to\infty}\lim_{n\to 0}\frac{1}{\beta N n}\log\int\cdots\int\left[\prod_{\alpha=1}^{n}\mathrm{d}\boldsymbol{\theta}^\alpha\right]\overline{\mathrm{e}^{-\beta\sum_\alpha H(\boldsymbol{\theta}^\alpha)}}$$

$$\overline{\mathrm{e}^{-\beta\sum_\alpha H(\boldsymbol{\theta}^\alpha)}} = \mathrm{e}^{\beta h\sum_{i\alpha}\cos(\theta_i^\alpha - \phi_i)}\prod_{i<j}\cosh\left[\frac{2\beta K}{\sqrt{N}}\sum_\alpha \cos\left(\theta_i^\alpha - \theta_j^\alpha + A^*\right)\right].$$

Expansion for large *N*, however, only leads to an expression in terms of the usual single-site replica order parameters if $\sin(2A^*) = 0$, i.e. for $A^* \in \left\{0, \frac{\pi}{2}\right\}$ ((2) and (3)). Here

$$\overline{f} = -\lim_{N\to 0}\frac{1}{n}\,\mathrm{extr}\{\Phi[\{\hat{\boldsymbol{q}}^{**}, \boldsymbol{q}^{**}\}] + \Psi[\{\hat{\boldsymbol{q}}^{**}\}]\} \tag{7}$$

$$\Phi[\cdots] = \frac{\mathrm{i}}{\beta}\sum_{**}\sum_{\alpha\beta}\hat{q}_{\alpha\beta}^{**}q_{\alpha\beta}^{**} + \beta K^2\cos^2(A^*)\sum_{**}\sum_{\alpha\beta}[q_{\alpha\beta}^{**}]^2$$
$$+ 2\beta K^2\sin^2(A^*)\sum_{\alpha\beta}\left[q_{\alpha\beta}^{ss}q_{\alpha\beta}^{cc} - q_{\alpha\beta}^{sc}q_{\alpha\beta}^{cs}\right] \tag{8}$$

$$\beta\Psi[\cdots] = \left\langle \log\int \mathrm{d}\boldsymbol{\theta}\, M(\boldsymbol{\theta}|\{\hat{\boldsymbol{q}}^{**}\})\right\rangle_\phi \tag{9}$$

with $(**) \in \{(cc), (ss), (sc), (cs)\}$, and with

$$M(\boldsymbol{\theta}|\{\hat{\boldsymbol{q}}^{**}\}) = \mathrm{e}^{\beta h\sum_\alpha\cos(\theta^\alpha - \phi) - \mathrm{i}\sum_{\alpha\beta}[\hat{q}_{\alpha\beta}^{cc}\cos(\theta_\alpha)\cos(\theta_\beta) + \hat{q}_{\alpha\beta}^{cs}\cos(\theta_\alpha)\sin(\theta_\beta)]}$$
$$\times\,\mathrm{e}^{-\mathrm{i}\sum_{\alpha\beta}[\hat{q}_{\alpha\beta}^{ss}\sin(\theta_\alpha)\sin(\theta_\beta) + \hat{q}_{\alpha\beta}^{sc}\sin(\theta_\alpha)\cos(\theta_\beta)]}. \tag{10}$$

[3] The transformation (4) implies $\phi_i \to \phi_i + \lambda + \tau_i\frac{\pi}{2}$, where $\tau_i \in \{-1, 1\}$. The variables λ and $\{\tau_i\}$ can be gauged away by putting $\theta_i \to \theta_i + \lambda + \tau_i\frac{\pi}{2}$ in combination with $J_{ij} \to J_{ij}\cos(\frac{\pi}{2}(\tau_i - \tau_j))$.

3.2. Replica-symmetric solutions

We now make the replica-symmetric (or ergodic) ansatz (RS) [30] for the saddle points. Our models (2) and (3) are then both found to be described by five remaining order parameters $\{Q_{cc}, Q_{cs}, q_{cc}, q_{ss}, q_{cs}\}$, and effective single spin measures $M(\theta|x, y, u, v, \phi)$:

$$M_{\rm I}(\theta|x, y, u, v, \phi) = {\rm e}^{\beta h\cos(\theta-\phi)+(\beta K)^2[[2Q_{cc}-1+q_{ss}-q_{cc}]\cos(2\theta)+2(Q_{cs}-q_{cs})\sin(2\theta)]} \times {\rm e}^{\beta K\cos(\theta)\sqrt{2}[x\sqrt{2}\sqrt{q_{cc}}+(u-{\rm i}v)\sqrt{q_{cs}}]+\beta K\sin(\theta)\sqrt{2}[y\sqrt{2}\sqrt{q_{ss}}+(u+{\rm i}v)\sqrt{q_{cs}}]} \tag{11}$$

$$M_{\rm II}(\theta|x, y, u, v, \phi) = {\rm e}^{\beta h\cos(\theta-\phi)-(\beta K)^2[[2Q_{cc}-1+q_{ss}-q_{cc}]\cos(2\theta)+2(Q_{cs}-q_{cs})\sin(2\theta)]} \times {\rm e}^{\beta K\cos(\theta)\sqrt{2}[x\sqrt{2}\sqrt{q_{ss}}+({\rm i}u+v)\sqrt{q_{cs}}]+\beta K\sin(\theta)\sqrt{2}[y\sqrt{2}\sqrt{q_{cc}}+({\rm i}u-v)\sqrt{q_{cs}}]} \tag{12}$$

(apart, in both cases, from a prefactor ${\rm e}^{(\beta K)^2[1-q_{cc}-q_{ss}]}$). We abbreviate

$$\langle\!\langle\cdots\rangle\!\rangle = \int Dx\,Dy\,Du\,Dv\cdots \qquad \langle f(\theta)\rangle_* = \frac{\int {\rm d}\theta f(\theta)M(\theta|x, y, u, v, \phi)}{\int {\rm d}\theta M(\theta|x, y, u, v, \phi)}$$

with the Gaussian measure $Dx = (2\pi)^{-\frac{1}{2}}\,{\rm e}^{-\frac{1}{2}x^2}$. The five order parameters are to be solved from the following saddle-point equations:

$$Q_{cc} = \langle\langle\!\langle\langle\cos^2(\theta)\rangle_*\rangle\!\rangle\rangle_\phi \qquad Q_{cs} = \langle\langle\!\langle\langle\cos(\theta)\sin(\theta)\rangle_*\rangle\!\rangle\rangle_\phi \tag{13}$$

$$q_{cc} = \langle\langle\!\langle\langle\cos(\theta)\rangle_*^2\rangle\!\rangle\rangle_\phi \qquad q_{ss}\langle\langle\!\langle\langle\sin(\theta)\rangle_*^2\rangle\!\rangle\rangle_\phi \tag{14}$$

$$q_{cs} = \langle\langle\!\langle\langle\cos(\theta)\rangle_*\langle\sin(\theta)\rangle_*\rangle\!\rangle\rangle_\phi. \tag{15}$$

Their physical meaning is

$$Q_{cc} = \lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle\cos^2(\theta_i)\rangle} \qquad Q_{cs} = \lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle\cos(\theta_i)\sin(\theta_i)\rangle} \tag{16}$$

$$q_{cc} = \lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle\cos(\theta_i)\rangle^2} \qquad q_{ss} = \lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle\sin(\theta_i)\rangle^2} \tag{17}$$

$$q_{cs} = \lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle\cos(\theta_i)\rangle\langle\sin(\theta_i)\rangle}. \tag{18}$$

To compactify future notation we define the following shorthand

$$\gamma_{cc} = \langle\cos^2(\theta)\rangle_* - \langle\cos(\theta)\rangle_*^2 \qquad \gamma_{ss} = \langle\sin^2(\theta)\rangle_* - \langle\sin(\theta)\rangle_*^2 \tag{19}$$

$$\gamma_{cs} = \langle\cos(\theta)\sin(\theta)\rangle_* - \langle\cos(\theta)\rangle_*\langle\sin(\theta)\rangle_*. \tag{20}$$

Similarly we can work out the RS free energy per oscillator, by insertion of the RS ansatz into (7). This results in

$$\overline{f} = {\rm extr}_{\{Q_{**},q_{**}\}} f[\{Q_{**}, q_{**}\}] \tag{21}$$

$$f[\cdots] = \beta K^2 U - \frac{1}{\beta}\int Dx\,Dy\,Du\,Dv\left\langle\log\int {\rm d}\theta\, M(\theta|x, y, u, v, \phi)\right\rangle_\phi \tag{22}$$

with the measures (11) and (12), and with

$$U_{\rm I} = 2Q_{cc}(Q_{cc}-1) + 2Q_{cs}^2 - q_{cc}(q_{cc}-1) - q_{ss}(q_{ss}-1) - 2q_{cs}^2 \tag{23}$$

$$U_{\rm II} = -2Q_{cc}(Q_{cc}-1) - 1 - 2Q_{cs}^2 + q_{cc} + q_{ss} - 2q_{cc}q_{ss} + 2q_{cs}^2. \tag{24}$$

The nature of the extremum in (21) follows from the high-temperature state. Expanding for $\beta \to 0$ gives $f[\cdots] = -\frac{1}{\beta}\log(2\pi) - \frac{1}{4}\beta h^2 + 2\beta K^2\,\mathrm{extr}\{\tilde{U}\} + \mathcal{O}(\beta^2)$, with

$$\tilde{U}_{\rm I} = Q_{cc}^2 - Q_{cc} + Q_{cs}^2 - \tfrac{1}{2}\left(q_{cc}^2 + q_{ss}^2\right) - q_{cs}^2$$
$$\tilde{U}_{\rm II} = -Q_{cc}^2 + Q_{cc} - \tfrac{1}{2} - Q_{cs}^2 - q_{cc}q_{ss} + q_{cs}^2.$$

For $\beta \to 0$ we must find the paramagnetic solution $Q_{cc} = \frac{1}{2}$ and $Q_{cs} = q_{cc} = q_{ss} = q_{cs} = 0$, so the nature of the desired extremum in (21) is

$$\text{I: min w.r.t. } \{Q_{cc}, Q_{cs}\}, \text{ max w.r.t. } \{q_{cc}, q_{ss}, q_{cs}\} \tag{25}$$

$$\text{II: min w.r.t. } \{q_{cs}, q_{cc} - q_{ss}\}, \text{ max w.r.t. } \{Q_{cc}, Q_{cs}, q_{cc} + q_{ss}\}. \tag{26}$$

Finally, for RS solutions we can also work out expression (6) for χ in the limit $N \to \infty$:

$$\frac{\chi_{\rm RS}}{\beta} = \frac{1}{2}(1 - q_{cc} - q_{ss}) - \left\langle\left\langle\left\langle \frac{1}{2}\cos(2\phi)(\gamma_{cc} - \gamma_{ss}) + \sin(2\phi)\gamma_{cs} \right\rangle\right\rangle\right\rangle_\phi$$
$$- \mathrm{i}\left\langle\left\langle\left\langle \cos(2\phi)\gamma_{cs} - \frac{1}{2}\sin(2\phi)(\gamma_{cc} - \gamma_{ss}) \right\rangle\right\rangle\right\rangle_\phi. \tag{27}$$

3.3. The AT lines

We determined the stability of our RS solution against 'replicon' perturbations: $q_{\alpha\beta}^{**} \to q_{\alpha\beta}^{**,\rm RS} + \eta_{\alpha\beta}^{**}$ with $\eta_{\alpha\beta}^{cc} = \eta_{\beta\alpha}^{cc}$ and $\eta_{\alpha\beta}^{ss} = \eta_{\beta\alpha}^{ss}$ (for all α, β), $\eta_{\alpha\alpha}^{**} = 0$ for all α, $\sum_\alpha \eta_{\alpha\beta}^{**} = \sum_\alpha \eta_{\beta\alpha}^{**} = 0$ for all α, and with all $|\eta_{\alpha\beta}^{**}| \ll 1$. Details are given in appendix B. Our calculation is more complicated than that in e.g. [25, 27], because, due to the pinning fields, we cannot generally use rotational symmetry; replicon fluctuations can have a more complicated structure. We found two types of AT instabilities (the physical RSB transition is the one occurring at the highest temperature). The first is

$$\text{models I and II: } (T/2K)^2 = \left\langle\left\langle\left\langle \gamma_{cc}\gamma_{ss} - \gamma_{cs}^2 \right\rangle\right\rangle\right\rangle_\phi. \tag{28}$$

The second AT instability is found to be model dependent:

$$\text{model I: } \det[\boldsymbol{E} - (T/2K)^2\boldsymbol{I}] = 0 \tag{29}$$

$$\text{model II: } \det[\boldsymbol{E} - (T/2K)^2\boldsymbol{C}] = 0 \tag{30}$$

with

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{31}$$

$$\boldsymbol{E} = \begin{pmatrix} \left\langle\left\langle\left\langle \frac{1}{2}\left(\gamma_{cc}^2 + \gamma_{ss}^2\right) + \gamma_{cs}^2 \right\rangle\right\rangle\right\rangle_\phi & \left\langle\left\langle\left\langle \frac{1}{2}\left(\gamma_{cc}^2 - \gamma_{ss}^2\right) \right\rangle\right\rangle\right\rangle_\phi & \langle\langle\langle \gamma_{cs}(\gamma_{cc} + \gamma_{ss}) \rangle\rangle\rangle_\phi \\ \left\langle\left\langle\left\langle \frac{1}{2}\left(\gamma_{cc}^2 - \gamma_{ss}^2\right) \right\rangle\right\rangle\right\rangle_\phi & \left\langle\left\langle\left\langle \frac{1}{2}\left(\gamma_{cc}^2 + \gamma_{ss}^2\right) - \gamma_{cs}^2 \right\rangle\right\rangle\right\rangle_\phi & \langle\langle\langle \gamma_{cs}(\gamma_{cc} - \gamma_{ss}) \rangle\rangle\rangle_\phi \\ \langle\langle\langle \gamma_{cs}(\gamma_{cc} + \gamma_{ss}) \rangle\rangle\rangle_\phi & \langle\langle\langle \gamma_{cs}(\gamma_{cc} - \gamma_{ss}) \rangle\rangle\rangle_\phi & \left\langle\left\langle\left\langle \gamma_{cc}\gamma_{ss} + \gamma_{cs}^2 \right\rangle\right\rangle\right\rangle_\phi \end{pmatrix}. \tag{32}$$

4. States with global reflection symmetry

We inspect the effect of global reflection $\theta_i \to -\theta_i$ on the order parameters (within the RS ansatz) with the help of (16)–(18):

$$\theta_i' = -\theta_i \quad \text{for all } i: \quad \begin{array}{lll} Q_{cc}' = Q_{cc} & Q_{cs}' = -Q_{cs} & \\ q_{cc}' = q_{cc} & q_{cs}' = -q_{cs} & q_{ss}' = q_{ss} \end{array}. \tag{33}$$

Invariance under this transformation implies $Q_{cs} = q_{cs} = 0$.

4.1. Implications for free energy and order parameters

For reflection-symmetric states the Gaussian variables $\{u, v\}$ disappear from the problem, and the measures (11) and (12) simplify to

$$M_{\mathrm{I}}(\theta|x, y, \phi) = \mathrm{e}^{\beta h \cos(\theta-\phi)-(\beta K)^2[2Q_{cc}-1+q_{ss}-q_{cc}]\cos(2\theta)+2\beta K[\cos(\theta)x\sqrt{q_{cc}}+\sin(\theta)y\sqrt{q_{ss}}]} \tag{34}$$

$$M_{\mathrm{II}}(\theta|x, y, \phi) = \mathrm{e}^{\beta h \cos(\theta-\phi)-(\beta K)^2[2Q_{cc}-1+q_{ss}-q_{cc}]\cos(2\theta)+2\beta K[\cos(\theta)x\sqrt{q_{ss}}+\sin(\theta)y\sqrt{q_{cc}}]}. \tag{35}$$

These expressions show, in combination with $p(\phi) = p(-\phi)$, that for any set of functions $\{k_\ell\}$ (with $\ell = 0, 1, 2, \ldots$) one has

$$\left\langle\!\left\langle\!\left\langle k_0(-\phi)\prod_{\ell>0}\langle k_\ell(-\theta)\rangle_*\right\rangle\!\right\rangle\!\right\rangle_\phi = \left\langle\!\left\langle\!\left\langle k_0(\phi)\prod_{\ell>0}\langle k_\ell(\theta)\rangle_*\right\rangle\!\right\rangle\!\right\rangle_\phi. \tag{36}$$

One is left with three order parameter equations:

$$Q_{cc} = \langle\langle\langle\cos^2(\theta)\rangle_*\rangle\rangle_\phi \qquad q_{cc} = \langle\langle\langle\cos(\theta)\rangle_*^2\rangle\rangle_\phi \qquad q_{ss} = \langle\langle\langle\sin(\theta)\rangle_*^2\rangle\rangle_\phi \tag{37}$$

Upon inserting $Q_{cs} = q_{cs} = 0$ into (21), (23) and (24), and using a saddle-point argument as $\beta \to \infty$ in the entropic term, one obtains the RS ground state energies:

$$\lim_{\beta\to\infty} f_{\mathrm{I}}[\cdots]/\beta K^2 = q_{cc} + q_{ss} + 2Q_{cc}(Q_{cc}-1) - q_{cc}^2 - q_{ss}^2 - |2Q_{cc} - 1 + q_{ss} - q_{cc}|$$

$$\lim_{\beta\to\infty} f_{\mathrm{II}}[\cdots]/\beta K^2 = q_{cc} + q_{ss} - 2Q_{cc}(Q_{cc}-1) - 2q_{ss}q_{cc} - 1 - |2Q_{cc} - 1 + q_{ss} - q_{cc}|.$$

According to (25) and (26), we must, for model I, minimize with respect to Q_{cc} and maximize with respect to $\{q_{cc}, q_{ss}\}$, whereas for model II we must minimize with respect to $q_{cc} - q_{ss}$ and maximize with respect to $\{Q_{cc}, q_{cc} + q_{ss}\}$. For both models this leads to

$$q_{cc} + q_{ss} = 1 \qquad Q_{cc} = \tfrac{1}{2}[1 + q_{cc} - q_{ss}] \tag{38}$$

giving a finite ground state energy: $\mathrm{extr}_{Q_{cc},q_{ss},q_{cc}} \lim_{\beta\to\infty}(\beta K^2)^{-1} f[Q_{cc}, q_{ss}, q_{cc}] = 0$. To find the RS ground states of our two models from the family $q_{cc} + q_{ss} = 1$, one would have to inspect the next order in T. In addition we should obviously expect replica symmetry to be broken for low temperatures.

4.2. Implications for AT lines and RS susceptibility

Reflection symmetry also simplifies expressions (29) and (30) for the AT instabilities, due to $\langle\langle\gamma_{cc}\gamma_{cs}\rangle\rangle_\phi = \langle\langle\gamma_{ss}\gamma_{cs}\rangle\rangle_\phi = 0$. Upon rejecting solutions which are immediately seen not to give the highest transition temperature, the AT lines of our two models are found to be the solutions of the following equations, respectively:

$$\left[\frac{T}{2K}\right]^2 = \max\left\{\frac{1}{2}\langle\langle\langle\gamma_{cc}^2 + \gamma_{ss}^2\rangle\rangle\rangle_\phi + \sqrt{\langle\langle\langle\gamma_{cs}^2\rangle\rangle\rangle_\phi^2 + \frac{1}{4}\langle\langle\langle\gamma_{cc}^2 - \gamma_{ss}^2\rangle\rangle\rangle_\phi^2}, \langle\langle\langle\gamma_{cc}\gamma_{ss} + \gamma_{cs}^2\rangle\rangle\rangle_\phi\right\} \tag{39}$$

$$\left[\frac{T}{2K}\right]^2 = \max\left\{\langle\langle\langle\gamma_{cs}^2\rangle\rangle\rangle_\phi + \sqrt{\langle\langle\langle\gamma_{cc}^2\rangle\rangle\rangle_\phi\langle\langle\langle\gamma_{ss}^2\rangle\rangle\rangle_\phi}, \langle\langle\langle\gamma_{cc}\gamma_{ss} - \gamma_{cs}^2\rangle\rangle\rangle_\phi\right\}. \tag{40}$$

For both models the left of the two arguments in the corresponding extremization problems give the required maximum. To see this, one first notes that

$$\langle\langle\langle\gamma_{cc}\gamma_{ss}\rangle\rangle\rangle_\phi = \tfrac{1}{2}\big\langle\big\langle\langle\gamma_{cc}^2 + \gamma_{ss}^2 - (\gamma_{cc} - \gamma_{ss})^2\rangle\big\rangle\big\rangle_\phi \leqslant \tfrac{1}{2}\big\langle\big\langle\langle\gamma_{cc}^2 + \gamma_{ss}^2\rangle\big\rangle\big\rangle_\phi \tag{41}$$

$$\langle\langle\langle\gamma_{cc}^2\rangle\rangle\rangle_\phi\langle\langle\langle\gamma_{ss}^2\rangle\rangle\rangle_\phi = \tfrac{1}{2}\left\{\langle\langle\langle\gamma_{cc}^2 + \gamma_{ss}^2\rangle\rangle\rangle_\phi^2 - \langle\langle\langle\gamma_{cc}^2\rangle\rangle\rangle_\phi^2 - \langle\langle\langle\gamma_{cc}^2\rangle\rangle\rangle_\phi^2\right\} \geqslant \tfrac{1}{2}\langle\langle\langle\gamma_{cc}^2 + \gamma_{ss}^2\rangle\rangle\rangle_\phi^2. \tag{42}$$

For model I we subtract the right argument of (39) from the left argument and find

$$\begin{aligned}\mathrm{LA} - \mathrm{RA} &= \tfrac{1}{2}\langle\langle\langle(\gamma_{cc} - \gamma_{ss})^2\rangle\rangle\rangle_\phi - \langle\langle\langle\gamma_{cs}^2\rangle\rangle\rangle_\phi + \sqrt{\langle\langle\langle\gamma_{cs}^2\rangle\rangle\rangle_\phi^2 + \tfrac{1}{4}\langle\langle\langle\gamma_{cc}^2 - \gamma_{ss}^2\rangle\rangle\rangle_\phi^2} \\ &\geqslant \tfrac{1}{2}\langle\langle\langle(\gamma_{cc} - \gamma_{ss})^2\rangle\rangle\rangle_\phi \geqslant 0.\end{aligned}$$

Thus the maximum in (39) is always realized by the left argument. Similarly, upon subtracting the right argument from the left argument in (40), using (41) and (42), we find

$$\begin{aligned}\mathrm{LA} - \mathrm{RA} &= 2\langle\langle\langle\gamma_{cs}^2\rangle\rangle\rangle_\phi + \sqrt{\langle\langle\langle\gamma_{cc}^2\rangle\rangle\rangle_\phi\langle\langle\langle\gamma_{ss}^2\rangle\rangle\rangle_\phi} - \langle\langle\langle\gamma_{cc}\gamma_{ss}\rangle\rangle\rangle_\phi \\ &\geqslant \sqrt{\langle\langle\langle\gamma_{cc}^2\rangle\rangle\rangle_\phi^2\langle\langle\langle\gamma_{ss}^2\rangle\rangle\rangle_\phi^2} - \tfrac{1}{2}\langle\langle\langle\gamma_{cc}^2 + \gamma_{ss}^2\rangle\rangle\rangle_\phi = \tfrac{1}{2}(\sqrt{2} - 1)\langle\langle\langle\gamma_{cc}^2 + \gamma_{ss}^2\rangle\rangle\rangle_\phi \geqslant 0.\end{aligned}$$

Hence also the maximum in (40) is realized by the left argument. We conclude that

$$\text{AT line I:}\quad \left[\frac{T}{2K}\right]^2 = \frac{1}{2}\langle\langle\langle\gamma_{cc}^2 + \gamma_{ss}^2\rangle\rangle\rangle_\phi + \sqrt{\langle\langle\langle\gamma_{cs}^2\rangle\rangle\rangle_\phi^2 + \frac{1}{4}\langle\langle\langle\gamma_{cc}^2 - \gamma_{ss}^2\rangle\rangle\rangle_\phi^2} \tag{43}$$

$$\text{AT line II:}\quad \left[\frac{T}{2K}\right]^2 = \langle\langle\langle\gamma_{cs}^2\rangle\rangle\rangle_\phi + \sqrt{\langle\langle\langle\gamma_{cc}^2\rangle\rangle\rangle_\phi\langle\langle\langle\gamma_{ss}^2\rangle\rangle\rangle_\phi}. \tag{44}$$

Finally, in reflection symmetric states with $p(\phi) = p(-\phi)\ \forall\phi$, where we may use identity (36), the replica-symmetric susceptibility (27) is found to be purely real:

$$\frac{\chi_{\mathrm{RS}}}{\beta} = \frac{1}{2}(1 - q_{cc} - q_{ss}) - \left\langle\!\!\left\langle\!\!\left\langle\frac{1}{2}\cos(2\phi)(\gamma_{cc} - \gamma_{ss}) + \sin(2\phi)\gamma_{cs}\right\rangle\!\!\right\rangle\!\!\right\rangle_\phi. \tag{45}$$

4.3. Reflection symmetry breaking transitions

Assuming reflection symmetry breaking to happen via continuous bifurcations allows us to determine its occurrence by studying the relevant entries of the RS Hessian of the free energy. Such transitions occur when

$$\det\begin{vmatrix}\partial^2 f\left[Q_{cc,q_{ss},q_{cc}}\right]/\partial Q_{cs}^2 & \partial^2 f\left[Q_{cc,q_{ss},q_{cc}}\right]/\partial Q_{cs}\partial q_{cs} \\ \partial^2 f\left[Q_{cc,q_{ss},q_{cc}}\right]/\partial Q_{cs}\partial q_{cs} & \partial^2 f\left[Q_{cc,q_{ss},q_{cc}}\right]/\partial q_{cs}^2\end{vmatrix} = 0. \tag{46}$$

The matrix elements of the RS Hessian are given in appendix A. We deal with both models simultaneously upon defining the variable $\tau \in \{-1, 1\}$, where $\tau = 1$ for model I and $\tau = -1$ for model II. This allows us to write

$$\frac{1}{\beta K^2}\frac{\partial^2 f}{\partial Q_{cs}^2} = 4\tau - 8(\beta K)^2\lambda_1 \qquad \frac{1}{\beta K^2}\frac{\partial^2 f}{\partial q_{cs}^2} = -4\tau + 8(\beta K)^2\lambda_2$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f}{\partial Q_{cs}\partial q_{cs}} = 4(\beta K)^2\lambda_3$$

with

$$\lambda_1 = \frac{1}{2}\langle\langle\langle\sin^2(2\theta)\rangle_* - \langle\sin(2\theta)\rangle_*^2\rangle\rangle\rangle_\phi$$
$$\lambda_2 = \langle\langle\langle[\langle\sin^2(\theta)\rangle_* - \langle\sin(\theta)\rangle_*^2][\langle\cos^2(\theta)\rangle_* - 3\langle\cos(\theta)\rangle_*^2] + [\langle\cos^2(\theta)\rangle_* - \langle\cos(\theta)\rangle_*^2][\langle\sin^2(\theta)\rangle_* - 3\langle\sin(\theta)\rangle_*^2] + 2[\langle\sin(\theta)\cos(\theta)\rangle_* - \langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*][\langle\sin(\theta)\cos(\theta)\rangle_* - 3\langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*]\rangle\rangle\rangle_\phi$$
$$\lambda_3 = 2\langle\langle\langle\sin(2\theta)\cos(\theta)\rangle_*\langle\sin(\theta)\rangle_* + \langle\sin(2\theta)\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_* - 2\langle\sin(2\theta)\rangle_*\langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*\rangle\rangle\rangle_\phi.$$

Insertion into (46) then reveals that reflection symmetry breaking transitions are marked by the highest temperature for which the following functions $\Sigma^{\text{ref}}_{\text{I,II}}(T)$ become negative:

$$\text{model I:}\quad \Sigma^{\text{ref}}_{\text{I}}(T) = (T/K)^2 - \sqrt{(\lambda_1-\lambda_2)^2-\lambda_3^2} - \lambda_1 - \lambda_2 \tag{47}$$

$$\text{model II:}\quad \Sigma^{\text{ref}}_{\text{II}}(T) = (T/K)^2 - \sqrt{(\lambda_1-\lambda_2)^2-\lambda_3^2} + \lambda_1 + \lambda_2. \tag{48}$$

Since the $\{\lambda_i\}$ are bounded, expressions (47) and (48) confirm that reflection symmetry will always be stable for sufficiently high temperatures.

5. States with global rotation symmetry

For uniformly distributed pinning angles, $p(\phi) = (2\pi)^{-1}$, we expect the macroscopic state to have global rotation symmetry for high temperatures. We inspect the effect of arbitrary global rotations $\theta_i \to \theta_i + \psi$ on the RS order parameters, using (16)–(18):

$$\theta_i' = \theta_i + \psi \text{ for all } i: q_{cc}' + q_{ss}' = q_{cc} + q_{ss}$$
$$\begin{pmatrix} Q_{cc}' - \frac{1}{2} \\ Q_{cs}' \end{pmatrix} = \begin{pmatrix} \cos(2\psi) & -\sin(2\psi) \\ \sin(2\psi) & \cos(2\psi) \end{pmatrix}\begin{pmatrix} Q_{cc} - \frac{1}{2} \\ Q_{cs} \end{pmatrix} \tag{49}$$
$$\begin{pmatrix} \frac{1}{2}(q_{cc}' - q_{ss}') \\ q_{cs}' \end{pmatrix} = \begin{pmatrix} \cos(2\psi) & -\sin(2\psi) \\ \sin(2\psi) & \cos(2\psi) \end{pmatrix}\begin{pmatrix} \frac{1}{2}(q_{cc} - q_{ss}) \\ q_{cs} \end{pmatrix}.$$

Invariance implies $Q_{cc} = \frac{1}{2}$, $Q_{cs} = q_{cs} = 0$, $q_{cc} = q_{ss} = q$, leaving just one order parameter. The invariant manifolds in order parameter space are

$$\left(Q_{cc} - \tfrac{1}{2}\right)^2 + Q_{cs}^2 = \epsilon_1 \qquad \tfrac{1}{4}(q_{cc} - q_{ss})^2 + q_{cs}^2 = \epsilon_2 \tag{50}$$

with the invariant state corresponding to $\epsilon_1 = \epsilon_2 = 0$. We will also find instances of rotation-invariant states without the pinning field distribution having rotational symmetry. Insertion of the ansatz $\left\{Q_{cc} = \frac{1}{2}, Q_{cs} = q_{cs} = 0, q_{cc} = q_{ss} = q\right\}$ into our RS equations reveals that (for nonzero h) the following condition is necessary and sufficient for the existence of a rotation-invariant solution:

$$\langle\cos^2(\phi)\rangle_\phi = \langle\sin^2(\phi)\rangle_\phi. \tag{51}$$

Hence, unless explicitly stated we will in this section not assume $p(\phi)$ to be uniform, but rely only on the two properties $p(\phi) = p(-\phi)$ (assumed to hold throughout this paper) and $\langle\cos(2\phi)\rangle_\phi = 0$ (to guarantee the existence of a rotation-invariant state).

5.1. Implications for free energy and order parameters

For rotation-symmetric states the measures (34) and (35) become identical:

$$M(\theta|x,y,\phi) = \mathrm{e}^{\beta h\cos(\theta-\phi)+2\beta K\sqrt{q}[x\cos(\theta)+y\sin(\theta)]}. \tag{52}$$

One can eliminate ϕ from the measure (52), by rotation of the Gaussian variables (x, y), leading to the following identity for any set of functions $\{k_\ell\}$ (with $\ell = 0, 1, 2, \ldots$):

$$\left\langle\left\langle\left\langle k_0(\phi)\prod_{\ell>0}\langle k_\ell(\theta)\rangle_*\right\rangle\right\rangle\right\rangle_\phi = \left\langle\left\langle\left\langle k_0(\theta)\prod_{\ell>0}\langle k_\ell(\theta+\phi)\rangle_{\mathrm{o}}\right\rangle\right\rangle\right\rangle_\phi \tag{53}$$

where $\langle\cdots\rangle_{\mathrm{o}}$ refers to averages calculated with the ϕ-independent measure $M(\theta|x,y) = M(\theta|x,y,0)$. As a consequence our subsequent calculations for rotation-invariant states will repeatedly involve various derivatives of the following generating function:

$$Z[x,y] = \log\int \mathrm{d}\theta\, \mathrm{e}^{\beta h\cos(\theta)+2\beta K\sqrt{q}[x\cos(\theta)+y\sin(\theta)]} = \log(2\pi) + \log I_0[\Xi] \tag{54}$$

with the shorthand $\Xi = \beta\sqrt{(h+2Kx\sqrt{q})^2 + (2Ky\sqrt{q})^2}$, and in which $I_n[z]$ denotes the nth modified Bessel function [31]. For instance:

$$\frac{1}{2\beta K\sqrt{q}}\frac{\partial}{\partial x}Z[x,y] = \langle\cos(\theta)\rangle_{\mathrm{o}} \tag{55}$$

$$\frac{1}{2\beta K\sqrt{q}}\frac{\partial}{\partial y}Z[x,y] = \langle\sin(\theta)\rangle_{\mathrm{o}} \tag{56}$$

$$\frac{1}{(2\beta K\sqrt{q})^2}\frac{\partial^2}{\partial x^2}Z[x,y] = \langle\cos^2(\theta)\rangle_{\mathrm{o}} - \langle\cos(\theta)\rangle_{\mathrm{o}}^2 \tag{57}$$

$$\frac{1}{(2\beta K\sqrt{q})^2}\frac{\partial^2}{\partial y^2}Z[x,y] = \langle\sin^2(\theta)\rangle_{\mathrm{o}} - \langle\sin(\theta)\rangle_{\mathrm{o}}^2 \tag{58}$$

$$\frac{1}{(2\beta K\sqrt{q})^2}\frac{\partial^2}{\partial x\partial y}Z[x,y] = \langle\sin(\theta)\cos(\theta)\rangle_{\mathrm{o}} - \langle\sin(\theta)\rangle_{\mathrm{o}}\langle\cos(\theta)\rangle_{\mathrm{o}} \tag{59}$$

$$\begin{aligned}\frac{1}{(2\beta K\sqrt{q})^3}\frac{\partial^3}{\partial x^2\partial y}Z[x,y] &= \langle\sin(\theta)\cos^2(\theta)\rangle_{\mathrm{o}} - \langle\sin(\theta)\rangle_{\mathrm{o}}\langle\cos^2(\theta)\rangle_{\mathrm{o}}\\ &\quad - \langle\sin(2\theta)\rangle_{\mathrm{o}}\langle\cos(\theta)\rangle_{\mathrm{o}} + 2\langle\sin(\theta)\rangle_{\mathrm{o}}\langle\cos(\theta)\rangle_{\mathrm{o}}^2\end{aligned} \tag{60}$$

$$\begin{aligned}\frac{1}{(2\beta K\sqrt{q})^3}\frac{\partial^3}{\partial x\partial y^2}Z[x,y] &= \langle\sin^2(\theta)\cos(\theta)\rangle_{\mathrm{o}} - \langle\cos(\theta)\rangle_{\mathrm{o}}\langle\sin^2(\theta)\rangle_{\mathrm{o}}\\ &\quad - \langle\sin(2\theta)\rangle_{\mathrm{o}}\langle\sin(\theta)\rangle_{\mathrm{o}} + 2\langle\cos(\theta)\rangle_{\mathrm{o}}\langle\sin(\theta)\rangle_{\mathrm{o}}^2.\end{aligned} \tag{61}$$

In order to work out the remaining order parameter equation for q, we apply the identity (53) to $k_1(\theta) = k_2(\theta) = \cos(\theta)$ and to $k_1(\theta) = k_2(\theta) = \sin(\theta)$, giving

$$\langle\langle\langle\langle\cos(\theta)\rangle_*^2\rangle\rangle\rangle_\phi = \langle\cos^2(\phi)\rangle_\phi\langle\langle\langle\cos(\theta)\rangle_{\mathrm{o}}^2\rangle\rangle + \langle\sin^2(\phi)\rangle_\phi\langle\langle\langle\sin(\theta)\rangle_{\mathrm{o}}^2\rangle\rangle \tag{62}$$

$$\langle\langle\langle\langle\sin(\theta)\rangle_*^2\rangle\rangle\rangle_\phi = \langle\sin^2(\phi)\rangle_\phi\langle\langle\langle\cos(\theta)\rangle_{\mathrm{o}}^2\rangle\rangle + \langle\cos^2(\phi)\rangle_\phi\langle\langle\langle\sin(\theta)\rangle_{\mathrm{o}}^2\rangle\rangle \tag{63}$$

Thus the remaining order parameter q is, for both models, to be solved from

$$q = \frac{1}{2}\int Dx\,Dy\,\langle\langle\cos(\theta)\rangle_*^2 + \langle\sin(\theta)\rangle_*^2\rangle_\phi = \frac{1}{2}\int Dx\,Dy\big[\langle\cos(\theta)\rangle_{\mathrm{o}}^2 + \langle\sin(\theta)\rangle_{\mathrm{o}}^2\big]. \tag{64}$$

Using properties of modified Bessel functions, such as $\frac{\mathrm{d}}{\mathrm{d}z}I_n[z] = \frac{1}{2}(I_{n+1}[z] + I_{n-1}[z])$ and $I_n[z] = (z/2n)(I_{n-1}[z] - I_{n+1}[z])$, the two relevant expressions (55) and (56) give

$$\langle\cos(\theta)\rangle_{\mathrm{o}} = \frac{\beta(h+2Kx\sqrt{q})}{\Xi}\frac{I_1[\Xi]}{I_0[\Xi]} \qquad \langle\sin(\theta)\rangle_{\mathrm{o}} = \frac{\beta(2Ky\sqrt{q})}{\Xi}\frac{I_1[\Xi]}{I_0[\Xi]}.$$

After writing (x, y) in polar coordinates, one then finds (64) reducing to

$$q = \int_0^{\pi}\frac{\mathrm{d}\psi}{2\pi}\int_0^{\infty}\mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2}\left\{\frac{I_1[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]}\right\}^2 \tag{65}$$

$$\Xi(r,\psi) = \beta\sqrt{4qK^2r^2 + h^2 + 4Khr\sqrt{q}\cos(\psi)}. \tag{66}$$

Inserting $Q_{cs} = q_{cs} = 0$, $Q_{cc} = \frac{1}{2}$ and $q_{cc} = q_{ss} = q$ into (21), (23) and (24) shows that for both models the disorder-averaged free energy per spin equals $\overline{f} = \mathrm{extr}_q f[q] - \frac{1}{\beta}\log(2\pi)$, with

$$f[q] = -2\beta K^2\left(q - \frac{1}{2}\right)^2 - \frac{1}{\beta}\int_0^{\pi}\frac{\mathrm{d}\psi}{\pi}\int_0^{\infty}\mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2}\log I_0[\Xi(r,\psi)]. \tag{67}$$

According to (25) and (26) the function $f[q]$ in both cases is to be maximized, so the RS rotation-invariant ground state is $q = \frac{1}{2}$.

5.2. *Implications for covariances AT lines and RS susceptibility*

Various terms involving the covariances γ_{**} ((19) and (20)) can be simplified via (53). We define $\overline{\gamma}_{**} = \gamma_{**}|_{\phi=0}$ (i.e. as calculated with the $\phi = 0$ averages $\langle\cdots\rangle_{\mathrm{o}}$). Focusing on the terms occurring in (43, 44) and using (51):

$$\left\langle\langle\langle\gamma_{cc}^2\rangle\rangle\right\rangle_{\phi} = \left\langle\left\langle\tfrac{1}{4}(\overline{\gamma}_{cc}+\overline{\gamma}_{ss})^2 + \overline{\gamma}_{cs}^2\right\rangle\right\rangle + \langle\cos^2(2\phi)\rangle_{\phi}\left\langle\left\langle\tfrac{1}{4}(\overline{\gamma}_{cc}-\overline{\gamma}_{ss})^2 - \overline{\gamma}_{cs}^2\right\rangle\right\rangle \tag{68}$$

$$\left\langle\langle\langle\gamma_{ss}^2\rangle\rangle\right\rangle_{\phi} = \left\langle\left\langle\tfrac{1}{4}(\overline{\gamma}_{cc}+\overline{\gamma}_{ss})^2 + \overline{\gamma}_{cs}^2\right\rangle\right\rangle + \langle\cos^2(2\phi)\rangle_{\phi}\left\langle\left\langle\tfrac{1}{4}(\overline{\gamma}_{cc}-\overline{\gamma}_{ss})^2 - \overline{\gamma}_{cs}^2\right\rangle\right\rangle \tag{69}$$

$$\left\langle\langle\langle\gamma_{cs}^2\rangle\rangle\right\rangle_{\phi} = \langle\sin^2(2\phi)\rangle_{\phi}\left\langle\left\langle\tfrac{1}{4}(\overline{\gamma}_{cc}-\overline{\gamma}_{ss})^2\right\rangle\right\rangle + \langle\cos^2(2\phi)\rangle_{\phi}\left\langle\left\langle\overline{\gamma}_{cs}^2\right\rangle\right\rangle \tag{70}$$

$$\tfrac{1}{4}\langle\langle\langle(\gamma_{cc}-\gamma_{ss})^2\rangle\rangle\rangle_{\phi} = \langle\sin^2(2\phi)\rangle_{\phi}\left\langle\left\langle\overline{\gamma}_{cs}^2\right\rangle\right\rangle + \langle\cos^2(2\phi)\rangle_{\phi}\left\langle\left\langle\tfrac{1}{4}(\overline{\gamma}_{cc}-\overline{\gamma}_{ss})^2\right\rangle\right\rangle. \tag{71}$$

From (57)–(59), in turn, one obtains explicit expressions for the $\{\overline{\gamma}_{**}\}$:

$$\overline{\gamma}_{cc} = \frac{1}{2} - \frac{I_2[\Xi]}{2I_0[\Xi]} + \frac{\beta^2(h+2Kx\sqrt{q})^2}{\Xi^2}\left\{\frac{I_2[\Xi]}{I_0[\Xi]} - \frac{I_1^2[\Xi]}{I_0^2[\Xi]}\right\} \tag{72}$$

$$\overline{\gamma}_{ss} = \frac{1}{2} - \frac{I_2[\Xi]}{2I_0[\Xi]} + \frac{\beta^2(2Ky\sqrt{q})^2}{\Xi^2}\left\{\frac{I_2[\Xi]}{I_0[\Xi]} - \frac{I_1^2[\Xi]}{I_0^2[\Xi]}\right\} \tag{73}$$

$$\overline{\gamma}_{cs} = \frac{\beta^2(h+2Kx\sqrt{q})(2Ky\sqrt{q})}{\Xi^2}\left\{\frac{I_2[\Xi]}{I_0[\Xi]} - \frac{I_1^2[\Xi]}{I_0^2[\Xi]}\right\}. \tag{74}$$

Combination of the above results with (43) and (44) gives the equations for the AT lines. We note that for rotation invariant states the two expressions (43) and (44) become identical:

$$\left[\frac{T}{K}\right]^2 = \int Dx\,Dy\left\{\left[\frac{I_2[\Xi]}{I_0[\Xi]} - \frac{I_1^2[\Xi]}{I_0^2[\Xi]}\right]^2 + \left[1 - \frac{I_1^2[\Xi]}{I_0^2[\Xi]}\right]^2\right\}.$$

After transformation of the Gaussian variables to polar coordinates, this gives for the AT lines of our two models the appealing result

$$\left[\frac{T}{K}\right]^2 = \int_0^{\pi}\frac{\mathrm{d}\psi}{\pi}\int_0^{\infty}\mathrm{d}r\, r\,\mathrm{e}^{-\frac{1}{2}r^2}\left\{\left[\frac{I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]}-\frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right]^2 + \left[1-\frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right]^2\right\}. \tag{75}$$

In a similar manner we work out expression (45) for rotation-invariant states:

$$\begin{aligned}\frac{\chi_{\mathrm{RS}}}{\beta} &= \frac{1}{2}(1-2q) - \left\langle\!\!\left\langle\!\!\left\langle \frac{1}{2}\cos(2\phi)(\gamma_{cc}-\gamma_{ss}) + \sin(2\phi)\gamma_{cs}\right\rangle\!\!\right\rangle\!\!\right\rangle_{\phi} \\ &= \frac{1}{2} - q - \int_0^{\pi}\frac{\mathrm{d}\psi}{2\pi}\int_0^{\infty}\mathrm{d}r\, r\,\mathrm{e}^{-\frac{1}{2}r^2}\left[\frac{I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]}-\frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right] \\ &\quad\times\left[\frac{h^2+4K^2r^2q\cos(2\psi)+4hKr\sqrt{q}\cos(\psi)}{h^2+4K^2r^2q+4hKr\sqrt{q}\cos(\psi)}\right]\end{aligned} \tag{76}$$

with the abbreviation (66). Note that $\lim_{h\to 0}\Xi(r,\psi) = 2\beta K r\sqrt{q}$ (i.e. independent of ψ), so that for rotation invariant states one has

$$\lim_{h\to 0}\chi_{\mathrm{RS}} = \beta\left(\tfrac{1}{2}-q\right). \tag{77}$$

5.3. Rotation symmetry breaking transitions

According to (50) we can study rotation symmetry breaking bifurcations most conveniently after the following transformation of the order parameters:

$$\begin{aligned}&Q_{cc} = \tfrac{1}{2}+\epsilon_1\cos(\omega_1) \qquad Q_{cs} = \epsilon_1\sin(\omega_1)\\ &q_{cc} = q+\epsilon_2\cos(\omega_2) \qquad q_{ss} = q-\epsilon_2\cos(\omega_2) \qquad q_{cs} = \epsilon_2\sin(\omega_2).\end{aligned}$$

Reflection symmetry breaking transitions are already studied ((47) and (48)), so we restrict ourselves to rotation symmetry breaking transitions which preserve reflection symmetry:

$$Q_{cc} = \tfrac{1}{2}+\epsilon_1 \qquad Q_{cs} = q_{cs} = 0 \qquad q_{cc} = q+\epsilon_2 \qquad q_{ss} = q-\epsilon_2.$$

Our interest is in continuous bifurcations of $\epsilon_1 \neq 0$ and/or $\epsilon_2 \neq 0$. We expand (21) around the rotation-invariant solution, using the second order derivatives in appendix A. Due to reflection symmetry, only second derivatives with an even total number of 's' subscripts in the corresponding order parameters can be nonzero. This gives for $0 \ll \epsilon_1, \epsilon_2 \ll 1$ (with $f_{\mathrm{RI}}[\cdots]$ denoting the free energy of the rotation invariant state):

$$\begin{aligned}f[\cdots] - f_{\mathrm{RI}}[\cdots] &= \frac{1}{2}\epsilon_1^2\frac{\partial^2 f}{\partial Q_{cc}^2} + \frac{1}{2}\epsilon_2^2\left\{\frac{\partial^2 f}{\partial q_{cc}^2}+\frac{\partial^2 f}{\partial q_{ss}^2} - 2\frac{\partial^2 f}{\partial q_{cc}\partial q_{ss}}\right\}\\ &\quad + \epsilon_1\epsilon_2\left\{\frac{\partial^2 f}{\partial Q_{cc}\partial q_{cc}} - \frac{\partial^2 f}{\partial Q_{cc}\partial q_{ss}}\right\} + \mathcal{O}(\epsilon^3).\end{aligned}$$

Hence the continuous rotation symmetry breaking transitions are marked by

$$\det\begin{vmatrix}\partial^2 f/\partial Q_{cc}^2 & \partial^2 f/\partial Q_{cc}\partial q_{cc} - \partial^2 f/\partial Q_{cc}\partial q_{ss}\\ \partial^2 f/\partial Q_{cc}\partial q_{cc} - \partial^2 f/\partial Q_{cc}\partial q_{ss} & \partial^2 f/\partial q_{cc}^2 + \partial^2 f/\partial q_{ss}^2 - 2\partial^2 f/\partial q_{cc}\partial q_{ss}\end{vmatrix} = 0. \tag{78}$$

We work out the second derivatives for rotation-invariant states, using (53). Given $p(\theta) = p(-\theta)$ and $\langle\cos(2\phi)\rangle_\phi$, these are found to depend on the pinning field distribution onlv through $\langle\cos(4\phi)\rangle_\phi$. Again we deal with both models simultaneously upon defining $\tau \in \{-1, 1\}$, where $\tau = 1$ for model I and $\tau = -1$ for model II:

$$\frac{1}{\beta K^2}\frac{\partial^2 f}{\partial Q_{cc}^2} = 4\tau - 8(\beta K)^2\rho_1 \tag{79}$$

$$\frac{1}{\beta K^2}\left\{\frac{\partial^2 f}{\partial q_{cc}^2} + \frac{\partial^2 f}{\partial q_{ss}^2} - 2\frac{\partial^2 f}{\partial q_{cc}\partial q_{ss}}\right\} = -4\tau + 8(\beta K)^2\rho_2 \tag{80}$$

$$\frac{1}{\beta K^2}\left\{\frac{\partial^2 f}{\partial Q_{cc}\partial q_{cc}} - \frac{\partial^2 f}{\partial Q_{cc}q_{ss}}\right\} = 4(\beta K)^2\rho_3 \tag{81}$$

where straightforward but tedious bookkeeping shows the quantities $\{\rho_1, \rho_2, \rho_3\}$ to be

$$\rho_1 = \tfrac{1}{4}\langle\!\langle 1 - \langle\cos(2\theta)\rangle_o^2 - \langle\sin(2\theta)\rangle_o^2\rangle\!\rangle + \tfrac{1}{4}\langle\cos(4\phi)\rangle_\phi\langle\!\langle\langle\cos(4\theta)\rangle_o - \langle\cos(2\theta\rangle_o^2 + \langle\sin(2\theta\rangle_o^2\rangle\!\rangle \tag{82}$$

$$\begin{aligned}\rho_2 = {}&\langle\!\langle\big[\langle\cos^2(\theta)\rangle_o - \langle\cos(\theta)\rangle_o^2\big]\big[\langle\cos^2(\theta)\rangle_o - 3\langle\cos(\theta)\rangle_o^2\big] + \big[\langle\sin^2(\theta)\rangle_o - \langle\sin(\theta)\rangle_o^2\big]\big[\langle\sin^2(\theta)\rangle_o\\ &- 3\langle\sin(\theta)\rangle_o^2\big] - \tfrac{1}{2}\big[\langle\cos(2\theta)\rangle_o - \langle\cos(\theta)\rangle_o^2 + \langle\sin(\theta)\rangle_o^2\big]\big[\langle\cos(2\theta)\rangle_o\\ &- 3\langle\cos(\theta)\rangle_o^2 + 3\langle\sin(\theta)\rangle_o^2\big]\rangle\!\rangle + \langle\cos(4\phi)\rangle_\phi\langle\!\langle\tfrac{1}{2}\big[\langle\cos(2\theta)\rangle_o - \langle\cos(\theta)\rangle_o^2\\ &+ \langle\sin(\theta)\rangle_o^2\big]\big[\langle\cos(2\theta)\rangle_o - 3\langle\cos(\theta)\rangle_o^2 + 3\langle\sin(\theta)\rangle_o^2\big] - 2[\langle\sin(\theta)\cos(\theta)\rangle_o\\ &- \langle\sin(\theta)\rangle_o\langle\cos(\theta)\rangle_o][\langle\sin(\theta)\cos(\theta)\rangle_o - 3\langle\sin(\theta)\rangle_o\langle\cos(\theta)\rangle_o]\rangle\!\rangle\end{aligned} \tag{83}$$

$$\begin{aligned}\rho_3 = {}&2\langle\!\langle\langle\sin^2(\theta)\rangle_o\langle\cos(\theta)\rangle_o^2 + \langle\cos^2(\theta)\rangle_o\langle\sin(\theta)\rangle_o^2 - \langle\sin(2\theta)\rangle_o\langle\sin(\theta)\rangle_o\langle\cos(\theta)\rangle_o\rangle\!\rangle\\ &+ 2\langle\cos(4\phi)\rangle_\phi\langle\!\langle\langle\sin^2(\theta)\rangle_o\langle\cos(\theta)\rangle_o^2 + \langle\cos^2(\theta)\rangle_o\langle\sin(\theta)\rangle_o^2\\ &+ \langle\sin(2\theta)\rangle_o\langle\sin(\theta)\rangle_o\langle\cos(\theta)\rangle_o - 2\langle\sin^2(\theta)\cos(\theta)\rangle_o\langle\cos(\theta)\rangle_o\\ &- 2\langle\cos^2(\theta)\sin(\theta)\rangle_o\langle\sin(\theta)\rangle_o\rangle\!\rangle.\end{aligned} \tag{84}$$

Inserting the second order derivatives into (78), using (82)–(84), shows that rotation symmetry breaking transitions which preserve reflection symmetry are marked by the highest temperature for which the following functions become negative:

$$\text{model I:}\quad \Sigma_{\rm I}^{\rm rot}(T) = (T/K)^2 - \sqrt{(\rho_1 - \rho_2)^2 - \rho_3^2} - \rho_1 - \rho_2 \tag{85}$$

$$\text{model II:}\quad \Sigma_{\rm II}^{\rm rot}(T) = (T/K)^2 - \sqrt{(\rho_1 - \rho_2)^2 - \rho_3^2} + \rho_1 + \rho_2. \tag{86}$$

What remains is to work out (82)–(84). For most terms we can use (54), (55)–(61) and (72)–(74). One further object in ρ_1, $\langle\!\langle\langle\cos(4\theta)\rangle_o\rangle\!\rangle$, can be calculated directly as follows:

$$\begin{aligned}\langle\!\langle\langle\cos(4\theta)\rangle_o\rangle\!\rangle &= \left\langle\!\!\left\langle\frac{\int d\theta\,\cos\left(4\theta + 4\,\text{atan}\left[\frac{2ky\sqrt{q}}{h+2Kx\sqrt{q}}\right]\right)e^{\Xi\cos(\theta)}}{2\pi I_0(\Xi)}\right\rangle\!\!\right\rangle\\ &= \left\langle\!\!\left\langle\left[1 - \frac{8\beta^4(2Ky\sqrt{q})^2(h+2Kx\sqrt{q})^2}{\Xi^4}\right]\frac{I_4[\Xi]}{I_0[\Xi]}\right\rangle\!\!\right\rangle.\end{aligned}$$

The final result of the exercise is, after transformation of (x, y) to polar coordinates,

$$\rho_1 = \frac{1}{4}\int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2}\int_0^\pi \frac{\mathrm{d}\psi}{\pi}\left\{1 - \frac{I_2^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\}$$
$$+\frac{1}{4}\langle\cos(4\phi)\rangle_\phi \int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2}\int_0^\pi \frac{\mathrm{d}\psi}{\pi}\left\{\frac{I_4[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]} - \frac{I_2^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\}$$
$$\times\left\{1 - \frac{8\beta^4[2Kr\sqrt{q}\sin(\psi)]^2[h+2Kr\sqrt{q}\cos(\psi)]^2}{\Xi^4(r,\psi)}\right\} \tag{87}$$

$$\rho_2 = \frac{1}{2}\int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2}\int_0^\pi \frac{\mathrm{d}\psi}{\pi}\left\{1 - \frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\}\left\{1 - \frac{3I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\} + \frac{1}{2}\langle\cos(4\phi)\rangle_\phi$$
$$\times\int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2}\int_0^\pi \frac{\mathrm{d}\psi}{\pi}\left\{\frac{I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]} - \frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\}\left\{\frac{I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]}\right.$$
$$\left. - \frac{3I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\}\left\{1 - \frac{8\beta^4[2Kr\sqrt{q}\sin(\psi)]^2[h+2Kr\sqrt{q}\cos(\psi)]^2}{\Xi^4(r,\psi)}\right\} \tag{88}$$

$$\rho_3 = \int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2}\int_0^\pi \frac{\mathrm{d}\psi}{\pi}\left\{1 - \frac{I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]}\right\}\frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]} + \langle\cos(4\phi)\rangle_\phi\int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2}$$
$$\times\int_0^\pi \frac{\mathrm{d}\psi}{\pi}\left\{1 - \frac{I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]}\right\}\left\{\frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]} - \frac{2I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]}\right\}$$
$$\times\left\{1 - \frac{8\beta^4[2Kr\sqrt{q}\sin(\psi)]^2[h+2Kr\sqrt{q}\cos(\psi)]^2}{\Xi^4(r,\psi)}\right\}. \tag{89}$$

The identity $I_{n+1}[z] = I_{n-1}[z] + nI_n[z](I_1[z] - I_0[z])/I_1[z]$ allows us to express any $I_{n>2}[z]$ in terms of the trio $\{I_0[z], I_1[z], I_2[z]\}$. Since the $\{\rho_i\}$ are bounded, expressions (85) and (86) confirm that, if rotation-invariant states exist (i.e. if $\langle\cos(2\phi)\rangle_\phi = \mathrm{O}$), rotation symmetry will be stable for sufficiently large temperatures.

To round off our discussion we return to reflection symmetry breaking transitions. We work out the $\{\lambda_i\}$ occurring in (47, 48) for rotation invariant states, and find

$$\lambda_1 = \frac{1}{4}\int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2}\int_0^\pi \frac{\mathrm{d}\psi}{\pi}\left\{1 - \frac{I_2^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\} - \frac{1}{4}\langle\cos(4\phi)\rangle_\phi\int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2}$$
$$\times\int_0^\pi \frac{\mathrm{d}\psi}{\pi}\left\{\frac{I_4[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]} - \frac{I_2^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\}$$
$$\times\left\{1 - \frac{8\beta^4[2Kr\sqrt{q}\sin(\psi)]^2[h+2Kr\sqrt{q}\cos(\psi)]^2}{\Xi^4(r,\psi)}\right\} \tag{90}$$

$$\lambda_2 = \frac{1}{2}\int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2}\int_0^\pi \frac{\mathrm{d}\psi}{\pi}\left\{1 - \frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\}\left\{1 - \frac{3I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\} - \frac{1}{2}\langle\cos(4\phi)\rangle_\phi$$
$$\times\int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2}\int_0^\pi \frac{\mathrm{d}\psi}{\pi}\left\{\frac{I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]} - \frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\}\left\{\frac{I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]}\right.$$
$$\left. - \frac{3I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\}\left\{1 - \frac{8\beta^4[2Kr\sqrt{q}\sin(\psi)]^2[h+2Kr\sqrt{q}\cos(\psi)]^2}{\Xi^4(r,\psi)}\right\} \tag{91}$$

$$\lambda_3 = \int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2} \int_0^\pi \frac{\mathrm{d}\psi}{\pi} \left\{1 - \frac{I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]}\right\} \frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]} - \langle\cos(4\phi)\rangle_\phi \int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2}$$
$$\times \int_0^\pi \frac{\mathrm{d}\psi}{\pi} \left\{1 - \frac{I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]}\right\} \left\{\frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]} - \frac{2I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]}\right\}$$
$$\times \left\{1 - \frac{8\beta^4[2Kr\sqrt{q}\sin(\psi)]^2[h + 2Kr\sqrt{q}\cos(\psi)]^2}{\Xi^4(r,\psi)}\right\}. \tag{92}$$

Apparently, for rotation-invariant states we obtain the trio $\{\lambda_1, \lambda_2, \lambda_3\}$ from $\{\rho_1, \rho_2, \rho_3\}$ by making in the latter the replacement $\langle\cos(4\phi)\rangle_\phi \to -\langle\cos(4\phi)\rangle_\phi$:

$$\lambda_i(\langle\cos(4\phi)\rangle_\phi) = \rho_i(-\langle\cos(4\phi)\rangle_\phi) \qquad i = 1, 2, 3. \tag{93}$$

The same is true for expressions (47) and (48) for the reflection symmetry breaking transition lines, which are obtained by substituting $\langle\cos(4\phi)\rangle_\phi \to -\langle\cos(4\phi)\rangle_\phi$ in the expressions (85) and (86) for rotation symmetry breaking transitions which preserve reflection symmetry.

6. Solution for specific choices for the pinning field statistics

We now turn to specific choices of the pinning field distribution $p(\phi)$, restricting ourselves to those with $p(\phi) = p(-\phi)$. We start with the benchmark case $h = 0$, absent pinning fields. Wherever possible we have tested our theoretical predictions against extensive numerical simulations. We iterated the Langevin dynamics corresponding to the measure (1) with a stochastic Euler method with time step $\Delta t = 10^{-3}$, for systems of size either $N = 400$ (with the advantage of better equilibration within experimentally accessible time scales) or $N = 800$ (with the advantage of reduced finite size effects). All simulation results shown are averages over ten experiments.

6.1. Absent pinning fields

For $h = 0$ the natural solution is the one with full rotational symmetry. Given the RS ansatz we are left with one order parameter, q, and (65) and (67) reduce to

$$q = \frac{1}{2} \int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2} \frac{I_1^2[\Xi(r)]}{I_0^2[\Xi(r)]} \tag{94}$$

$$\overline{f} = \max_q \left\{-2\beta K^2 \left(q - \frac{1}{2}\right)^2 - \frac{1}{\beta} \int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2} \log I_0[\Xi(r)] - \frac{1}{\beta} \log(2\pi)\right\} \tag{95}$$

with $\Xi(r) = 2\beta K\sqrt{q}r$. There is a second order transition at $T_c = K$ from a paramagnetic state ($q = 0$) to an ordered state ($q > 0$), and no evidence for first order transitions. As the temperature is lowered further, q increases monotonically towards its maximum value at $T = 0$, as $q = \frac{1}{2} - T\sqrt{\pi}/4K + \mathcal{O}(T^2)$. Close to the critical point we can expand q in powers of $\tau = 1 - T/T_c$ and find $q = \frac{1}{2}\tau + \mathcal{O}(\tau^2)(\tau \downarrow 0)$. All this agrees with the results obtained earlier for Gaussian interactions [32].

For $h = 0$ the RS susceptibility becomes $\chi_{RS} = \beta\left(\frac{1}{2} - q\right)$, according to (77), and obeys $\lim_{T\to 0} \chi_{RS} = \sqrt{\pi}/4K$ and $\chi_{RS} = 1/2T$ for $T \geqslant K$. Close to the critical point, expansion in $\tau = 1 - T/T_c$ reveals that $\chi_{RS} = 1/2K + \mathcal{O}(\tau^2)$. χ_{RS} thus increases from the value $\sqrt{\pi}/4K$ at $T = 0$ to a cusp with value $1/2K$ at $T > T_c$, followed by monotonic $1/2T$ decay to zero in the regime $T > T_c$ (see also figure 1). Expression (75) for the AT instability simplifies similarly for $h = 0$ to

$$\left[\frac{T}{K}\right]^2 = \int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2} \left\{\left[\frac{I_2[\Xi(r)]}{I_0[\Xi(r)]} - \frac{I_1^2[\Xi(r)]}{I_0^2[\Xi(r)]}\right]^2 + \left[1 - \frac{I_1^2[\Xi(r)]}{I_0^2[\Xi(r)]}\right]^2\right\}. \tag{96}$$

From this it follows that the AT instability occurs at $T_{\mathrm{AT}} = K$. In both models replica symmetry breaks as soon as we leave the paramagnetic region at $T = T_c = K$. Using the properties $I_n(z)/I_0(z) = 1 - n^2/2z + \mathcal{O}(z^{-2})$ ($|z| \to \infty$) of the modified Bessel functions [31], we can also study the behaviour of both sides of (96) for $T \to 0$. This reveals that the degree of RS instability diverges near $T = 0$: $\lim_{T\to 0}(\mathrm{RHS}/\mathrm{LHS}) = \int_0^\infty \mathrm{d}r\, r^{-1}\, \mathrm{e}^{-\frac{1}{2}r^2} = \infty$.

Finally we work out the functions (85) and (86) whose zeros mark rotation symmetry breaking. For $h = 0$ it follows from (93) that the two types of symmetry breaking coincide, i.e. $\Sigma^{\mathrm{rot}}_{\mathrm{I,II}} = \Sigma^{\mathrm{ref}}_{\mathrm{I,II}} = \Sigma_{\mathrm{I,II}}$. All ψ integrals are trivial, and (87)–(89) reduce to

$$\rho_1 = \frac{1}{4}\int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2} \left\{1 - \frac{I_2^2[\Xi(r)]}{I_0^2[\Xi(r)]}\right\} \tag{97}$$

$$\rho_2 = \frac{1}{2}\int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2} \left\{1 - \frac{I_1^2[\Xi(r)]}{I_0^2[\Xi(r)]}\right\}\left\{1 - \frac{3I_1^2[\Xi(r)]}{I_0^2[\Xi(r)]}\right\} \tag{98}$$

$$\rho_3 = \int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2} \left\{1 - \frac{I_2[\Xi(r)]}{I_0[\Xi(r)]}\right\}\frac{I_1^2[\Xi(r)]}{I_0^2[\Xi(r)]}. \tag{99}$$

For $T \geqslant T_c = K$ one simply obtains $(\rho_1, \rho_2, \rho_3) = \left(\frac{1}{4}, \frac{1}{2}, 0\right)$. Hence $\Sigma_{\mathrm{I}}(T \geqslant K) = (T/K)^2 - 1$ and $\Sigma_{\mathrm{II}}(T \geqslant K) = (T/K)^2 + \frac{1}{2}$. Near the critical point, expansion in $\tau = 1 - T/K$ reveals that $(\rho_1, \rho_2, \rho_3) = \left(\frac{1}{4}, \frac{1}{2} - 2\tau, \tau\right) + \mathcal{O}(\tau^{3/2})$ so

$$\tau = 1 - T/K\colon \Sigma_{\mathrm{I}}(T) = 2\tau + \mathcal{O}(\tau^{\frac{3}{2}}) \qquad \Sigma_{\mathrm{II}}(T) = \tfrac{3}{2} - 2\tau + \mathcal{O}(\tau^{\frac{3}{2}}).$$

Thus for $T > K$ the rotation-invariant RS state is stable (only marginal for model I at $T = T_c$), followed by restored stability for $T < T_c$. Near $T = 0$ one has $(\rho_1, \rho_2, \rho_3) = (T\sqrt{\pi}/2K)(1, -1, 2) + \mathcal{O}(T^2)$, so either $\Sigma_{\mathrm{I,II}}(T)$ does not exist, or $\Sigma_{\mathrm{I,II}}(T) = \mathcal{O}(T^2)$.

In figure 1 we show our results, together with those of homogeneously distributed pinning angles (see below), by plotting q and χ_{RS} as functions of temperature, and testing them against numerical simulations. We find reasonable agreement, given the CPU limitations on system size and equilibration times. In spite of the differences between models I and II at the microscopic level (notably in terms of frustration properties of spin loops), without pinning fields there is no macroscopic distinction between their physical behaviour in the replica-symmetric state, probably due to the overruling amount of frustration. The models have identical $q \neq 0$ and RSB transition lines and identical values of the replica-symmetric physical observables, with a paramagnetic state for $T > K$, and a spin-glass state for $T < K$. The only difference is the degree of stability of the rotation-invariant state against non-rotationally-invariant fluctuations, since $\Sigma_{\mathrm{I}}(T) \neq \Sigma_{\mathrm{II}}(T)$, which cannot be measured directly.

6.2. Homogeneously distributed pinning angles

Our second choice is the homogeneous distribution $p(\phi) = (2\pi)^{-1}$. The Hamiltonian (1) is no longer invariant under simultaneous rotation of all spins, but we still expect the macroscopic state of the system to have global rotation symmetry. We cannot simplify equations (65), (67) and (76) further:

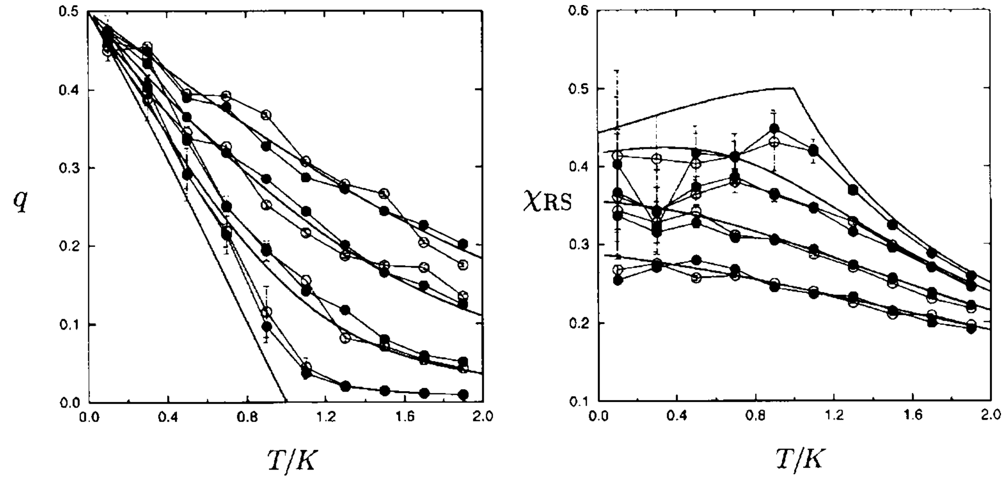

Figure 1. Theoretical predictions for the RS order parameter q (left) and the susceptibility χ_{RS} (right) as functions of temperature, for models I and II and with $K = 1$, in the cake of homogeneously distributed pinning angles. Different curves correspond to different values of the pinning field strength h, taken from $\{0, 1, 2, 3\}$ (lower to upper in left-hand picture, upper to lower in right-hand picture). Connected markers: results of numerical simulations with $N = 400$ (model I: ●, model II: ○).

$$q = \int_0^\pi \frac{\mathrm{d}\psi}{2\pi} \int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2} \left\{ \frac{I_1[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]} \right\}^2 \tag{100}$$

$$f[q] = -2\beta K^2 \left(q - \frac{1}{2}\right)^2 - \frac{1}{\beta} \int_0^\pi \frac{\mathrm{d}\psi}{\pi} \int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2} \log I_0[\Xi(r,\psi)] \tag{101}$$

$$\frac{\chi_{\mathrm{RS}}}{\beta} = \frac{1}{2} - q - \int_0^\pi \frac{\mathrm{d}\psi}{2\pi} \int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2} \left[\frac{I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]} - \frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]} \right] \times \left[\frac{h^2 + 4K^2r^2 q\cos(2\psi) + 4hKr\sqrt{q}\cos(\psi)}{h^2 + 4K^2r^2 q + 4hKr\sqrt{q}\cos(\psi)} \right] \tag{102}$$

with $\Xi(r,\psi) = \beta\sqrt{4qK^2r^2 + h^2 + 4Khr\sqrt{q}\cos(\psi)}$. For $h \neq 0$ one no longer expects to find a phase transition from a $q = 0$ to a $q > 0$ state; this is clear upon expanding (100) for $\beta \to 0$, giving $q = \frac{1}{8}(\beta h)^2 + \mathcal{O}(\beta^3)$. For weak fields one has $q = h^2/8(T^2 - K^2) + \mathcal{O}(h^3)$, in the regime $T > K$. For strong fields one finds $\lim_{h\to\infty} q = \frac{1}{2}$, $\lim_{h\to\infty}(\overline{f}/h) = -1$ and $\lim_{h\to\infty}\chi_{\mathrm{RS}} = 0$. For $T \to 0$, on the other hand, one finds

$$q = \frac{1}{2} - T \int_0^\pi \frac{\mathrm{d}\psi}{2\pi} \int_0^\infty \frac{\mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2}}{\sqrt{h^2 + 2K^2r^2 + 2Khr\sqrt{2}\cos(\psi)}} + \mathcal{O}(T^2) \tag{103}$$

$$\lim_{T\to 0} \chi_{\mathrm{RS}} = \int_0^\pi \frac{\mathrm{d}\psi}{\pi} \int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2} \frac{[h + \sqrt{2}Kr\cos(\psi)]^2}{[h^2 + 2K^2r^2 + 2Khr\sqrt{2}\cos(\psi)]^{3/2}}. \tag{104}$$

Away from $T, h \in \{0, \infty\}$ we must resort mainly to numerical evaluation of our equations. In figure 1 we compare the result of this exercise with numerical simulations, carried out for $N = 400$ and $h \in \{0, 1, 2, 3\}$. The agreement is satisfactory, apart from the $h = 0$ and low-temperature results, where finite size effects and equilibration problems are most

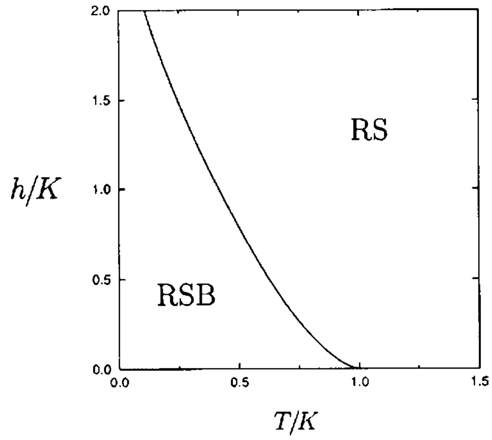

Figure 2. The phase diagram for models I and II, in the case of homogeneously distributed pinning angles. Solid: the AT instability, signalling a second order transition from a replica-symmetric paramagnetic state (RS) to a locally ordered state without replica symmetry (RSB), and possibly to states without global spherical symmetry. The transition field strength diverges as $T/K \to 0$.

prominent. We note, in comparison with the phase diagram of figure 2, that the most serious deviations, observed in the susceptibility curves, occur in the RSB region where χ_{RS} is indeed not expected to be correct.

Next we turn to expression (105) for the AT line:

$$\left[\frac{T}{K}\right]^2 = \int_0^\pi \frac{\mathrm{d}\psi}{\pi}\int_0^\infty \mathrm{d}r\, r\,\mathrm{e}^{-\frac{1}{2}r^2}\left\{\left[\frac{I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]} - \frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right]^2 + \left[1 - \frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right]^2\right\}. \tag{105}$$

For $h \to 0$ we recover (96) with replica instability for $T < K$. Expanding (105) for weak fields in the regime $T > K$ gives $(T/K)^2 = 1 - h^2/2(T^2 - K^2) + \mathcal{O}(h\sqrt{h})$, which has no solution; hence weak fields strengthen replica stability for $T > K$ (as expected). For $h \to \infty$ we find $\Xi(r,\psi) \to \beta h$, as a result of which we get, using $I_n(z)/I_0(z) = 1 - n^2/2z + \mathcal{O}(z^{-2})$ ($|z| \to \infty$), for the right-hand side of (105) the asymptotic form RHS $= T^2/2h^2 + \mathcal{O}(h^{-3})$ ($h \to \infty$), Hence for every nonzero temperature there is a pinning field strength above which replica symmetry holds. Investigation of the limit $T \to 0$ in (105) shows that we now end up with

$$\lim_{T\to 0}\frac{\mathrm{RHS}}{(T/K)^2} = \int_0^\pi \frac{\mathrm{d}\psi}{\pi}\int_0^\infty \frac{\mathrm{d}r\, r\,\mathrm{e}^{-\frac{1}{2}r^2}}{r^2 + \frac{1}{2}h^2/K^2 + \sqrt{2}rh\cos\psi/K}$$
$$= \mathrm{e}^{-\frac{1}{4}h^2/K^2}\int_0^\infty \frac{\mathrm{d}r}{r}\,\mathrm{e}^{-\frac{1}{2}r^2} I_0[hr/K\sqrt{2}] = \infty.$$

The field $h_{\mathrm{AT}}(T)$ which marks the AT instability diverges as $T/K \to 0$. For general fields and temperatures the integrals in (105) are evaluated numerically. This gives rise to the phase diagram shown in figure 2.

For homogeneously distributed pinning angles, where $\langle\cos(4\phi)\rangle_\phi = 0$, it follows from (93) that the two types of rotation symmetry breaking coincide: $\Sigma^{\mathrm{rot}}_{\mathrm{I,II}}(T) = \Sigma^{\mathrm{ref}}_{\mathrm{I,II}}(T) = \Sigma_{\mathrm{I,II}}(T)$. The constituents of (85) and (86) become

$$\rho_1 = \frac{1}{4}\int_0^\infty \mathrm{d}r\, r\,\mathrm{e}^{-\frac{1}{2}r^2}\int_0^\pi \frac{\mathrm{d}\psi}{\pi}\left\{1 - \frac{I_2^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\} \tag{106}$$

$$\rho_2 = \frac{1}{2}\int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2} \int_0^\pi \frac{\mathrm{d}\psi}{\pi} \left\{1 - \frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\}\left\{1 - \frac{3I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\} \tag{107}$$

$$\rho_3 = \int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2} \int_0^\pi \frac{\mathrm{d}\psi}{\pi} \left\{1 - \frac{I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]}\right\} \frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}. \tag{108}$$

Since $|I_n[z]/I_0[z]| \leqslant 1$ it is clear that $\rho_1 \in \left[0, \frac{1}{4}\right]$, $\rho_2 \in \left[-1, \frac{1}{2}\right]$ and $\rho_3 \in [0, 1]$. Hence, given the existence of the square root, one immediately obtains from (85) and (86) the bounds $\Sigma_{\mathrm{I}}(T) \geqslant (T/K)^2 - 1$ and $\Sigma_{\mathrm{II}}(T) \geqslant (T/K)^2 - \frac{3}{4}$. Thus the rotation-invariant state is stable for $T \geqslant K$ (model I) and $T > \frac{1}{2}\sqrt{3}$ (model II). For weak fields we may again, in the regime $T > K$, expand in powers of h. This gives

$$\Sigma_{\mathrm{I}}(T) = \left[\frac{T}{K}\right]^2 - 1 + \frac{h^2}{T^2 - K^2} + \mathcal{O}(h^3) \tag{109}$$

$$\Sigma_{\mathrm{II}}(T) = \left[\frac{T}{K}\right]^2 + \frac{1}{2} + \mathcal{O}(h^3). \tag{110}$$

Weak pinning fields stabilize the rotation-invariant state for model I, but do not affect the stability for model II. For strong pinning fields we expand in powers of h^{-1} and find $\Sigma_{\mathrm{I,II}}(T) = (T/K)^2 + \mathcal{O}(h^{-2})$, so rotation symmetry is stable for any temperature.

For arbitrary pinning field strengths and temperatures, the integrals in (106)–(108) are evaluated numerically. This reveals that for all nonzero temperatures and all field strengths $\Sigma_{\mathrm{I,II}}(T) \geqslant 0$, implying (in combination with the lack of evidence of first order transitions, and within the RS ansatz) the prediction that the system is always in a rotation-invariant state. There are no further RS transitions, and the phase diagram is given by figure 2. Again, models I and II differ only in the degree of stability of the rotation-invariant state against non-rotationally-invariant fluctuations.

6.3. Inhomogeneous distributions: $p(\phi) = \frac{1}{2}\delta[\phi - \alpha] + \frac{1}{2}\delta[\phi + \alpha]$

Finally we turn to pinning angle distributions of the form $p(\phi) = \frac{1}{2}\delta[\phi - \alpha] + \frac{1}{2}\delta[\phi + \alpha]$, with $\alpha \in [0, \pi]$. By varying α we can control this distribution to be either unimodal or bimodal. One now has $\langle\cos(2\phi)\rangle_\phi = \cos(2\alpha)$ and $\langle\cos(4\phi)\rangle_\phi = \cos(4\alpha)$. Hence, according to (51) we can have rotation-invariant solutions only when $\alpha \in \{\pi/4, 3\pi/4\}$; in the latter cases one has $\langle\cos(4\pi)\rangle_\phi = -1$, so as far as rotation-invariant solutions and their stability are concerned the $\alpha \in \{\pi/4, 3\pi/4\}$ models behave identically. For $\alpha \notin \{\pi/4, 3\pi/4\}$ there is generally only the overall reflection symmetry $\phi_i \to -\phi_i$ to be exploited, and we therefore must resort mainly to evaluating our equations numerically.

In view of the above discussion we concentrated on $\alpha \in \left\{0, \frac{\pi}{4}, \frac{\pi}{2}\right\}$, covering bimodal $\left(\alpha = \frac{\pi}{4}, \frac{\pi}{2}\right)$ and unimodal $(\alpha = 0)$ distributions, and models with rotation-invariant solutions $\left(\alpha = \frac{\pi}{4}\right)$ as well as cases without $\left(\alpha = 0, \frac{\pi}{2}\right)$. We tested our theory by comparison with numerical simulations. The general effect of random pinning fields is to break the symmetry between our models, and the values of otherwise identical order parameter pairs such as $\{q_{cc}, q_{ss}\}$ and $\{Q_{cc}, Q_{ss}\}$. The agreement between theory and simulations is generally satisfactory, except for the susceptibility when measured at low values of T and h, where one faces equilibration and replica symmetry breaking difficulties.

6.3.1. $p(\phi) = \frac{1}{2}\delta\left[\phi - \frac{1}{4}\pi\right] + \frac{1}{2}\delta\left[\phi + \frac{1}{4}\pi\right]$. For $\alpha = \pi/4$ one still has rotation-invariant solutions to our RS equations, and all results obtained earlier for $p(\phi) = (2\pi)^{-1}$ relating to

q (100), the free energy per oscillator (101), χ_{RS} (102) and the AT line (105) can be taken over, without alteration. The difference with $p(\phi) = (2\pi)^{-1}$ lies in the stability against fluctuations which violate rotational symmetry. Since here $\langle\cos(4\pi)\rangle_\phi = -1$, we now obtain the symmetry breaking transitions as the zeros of the following functions (using (93)):

$$\Sigma_{\mathrm{I}}^{\mathrm{rot}}(T) = (T/K)^2 - \sqrt{(\rho_1(-1) - \rho_2(-1))^2 - \rho_3^2(-1)} - \rho_1(-1) - \rho_2(-1) \tag{111}$$

$$\Sigma_{\mathrm{I}}^{\mathrm{ref}}(T) = (T/K)^2 - \sqrt{(\rho_1(+1) - \rho_2(+1))^2 - \rho_3^2(+1)} - \rho_1(+1) - \rho_2(+1) \tag{112}$$

$$\Sigma_{\mathrm{II}}^{\mathrm{rot}}(T) = (T/K)^2 - \sqrt{(\rho_1(-1) - \rho_2(-1))^2 - \rho_3^2(-1)} - \rho_1(-1) + \rho_2(-1) \tag{113}$$

$$\Sigma_{\mathrm{II}}^{\mathrm{ref}}(T) = (T/K)^2 - \sqrt{(\rho_1(+1) - \rho_2(+1))^2 - \rho_3^2(+1)} - \rho_1(+1) + \rho_2(+1) \tag{114}$$

with

$$\begin{aligned}\rho_1(\kappa) = {}& \frac{1}{4}\int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2} \int_0^\pi \frac{\mathrm{d}\psi}{\pi} \left\{1 - \frac{I_2^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\} \\ & + \frac{1}{4}\kappa \int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2} \int_0^\pi \frac{\mathrm{d}\psi}{\pi} \left\{\frac{I_4[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]} - \frac{I_2^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\} \\ & \times \left\{1 - \frac{8\beta^4[2Kr\sqrt{q}\sin(\psi)]^2[h + 2Kr\sqrt{q}\cos(\psi)]^2}{\Xi^4(r,\psi)}\right\}\end{aligned} \tag{115}$$

$$\begin{aligned}\rho_2(\kappa) = {}& \frac{1}{2}\int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2} \int_0^\pi \frac{\mathrm{d}\psi}{\pi} \left\{1 - \frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\}\left\{1 - \frac{3I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\} \\ & + \frac{1}{2}\kappa \int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2} \int_0^\pi \frac{\mathrm{d}\psi}{\pi} \left\{\frac{I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]} - \frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\}\left\{\frac{I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]}\right. \\ & \left. - \frac{3I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]}\right\}\left\{1 - \frac{8\beta^4[2Kr\sqrt{q}\sin(\psi)]^2[h + 2Kr\sqrt{q}\cos(\psi)]^2}{\Xi^4(r,\psi)}\right\}\end{aligned} \tag{116}$$

$$\begin{aligned}\rho_3(\kappa) = {}& \int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2} \int_0^\pi \frac{\mathrm{d}\psi}{\pi} \left\{1 - \frac{I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]}\right\}\frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]} \\ & + \kappa \int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\frac{1}{2}r^2} \int_0^\pi \frac{\mathrm{d}\psi}{\pi} \left\{1 - \frac{I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]}\right\}\left\{\frac{I_1^2[\Xi(r,\psi)]}{I_0^2[\Xi(r,\psi)]} - \frac{2I_2[\Xi(r,\psi)]}{I_0[\Xi(r,\psi)]}\right\} \\ & \times \left\{1 - \frac{8\beta^4[2Kr\sqrt{q}\sin(\psi)]^2[h + 2Kr\sqrt{q}\cos(\psi)]^2}{\Xi^4(r,\psi)}\right\}.\end{aligned} \tag{117}$$

The inequality $|I_n[z]/I_0[z]| \leqslant 1$ allows one to obtain the crude bound $T/K \leqslant 3$ for the zeros of (111)–(114). For strong fields one has

$$(\rho_1, \rho_2, \rho_3) = \frac{1-\kappa}{\beta h}(1, -1, 2) + \mathcal{O}(h^{-2}) \qquad (h \to \infty)$$

and thus $\Sigma_{\mathrm{I,II}}^{\mathrm{rot,ref}}(T) = (T/K)^2 + \mathcal{O}(h^{-2})$, confirming stability of the rotation-invariant state. For weak fields we may again for $T > K$ expand in powers of h. This reveals that those terms in $\{\rho_1, \rho_2, \rho_3\}$ which are proportional to κ are of subleading order $\mathcal{O}(h^{-3})$ as $h \to 0$. Hence we revert back to expressions (109) and (110). Apparently, the effect of switching on the pinning fields is again to stabilize the rotation-invariant state.

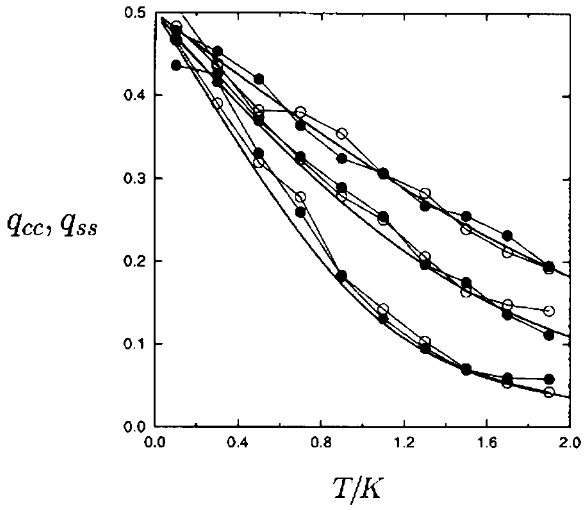

Figure 3. Theoretical predictions for the RS order parameters q_{cc} and q_{ss} as functions of temperature, for model I with $K = 1$, in the case of pinning angle distribution $p(\phi) = \frac{1}{2}\delta[\phi - \frac{1}{4}\pi] + \frac{1}{2}\delta[\phi + \frac{1}{4}\pi]$ (where the theory predicts that $q_{cc} = q_{ss}$). Different curves correspond to different values of the pinning field strength h, taken from $\{1, 2, 3\}$ (lower to upper). Markers: results of numerical simulations with $N = 400$ (q_{cc}: ●, q_{ss}: ○). Error bars in the measurements are of the order of 0.02; the deviations observed are mainly finite size effects. Note that the property $q_{cc} = q_{ss}$ is the fingerprint of rotation-invariant states.

Numerical evaluation of (111)–(114) showed that for all nonzero temperatures and field strengths $\Sigma^{\text{rot,ref}}_{\text{I,II}}(T) \geqslant 0$. This implies (in combination with the lack of evidence of first order transitions, and within the RS ansatz) the nontrivial prediction that for $p(\phi) = \frac{1}{2}\delta\big[\phi - \frac{1}{4}\pi\big] + \frac{1}{2}\delta\big[\phi + \frac{1}{4}\pi\big]$, which is a distribution *without* rotation invariance, the system is still *always* in a rotation-invariant state. This statement and its quantitative consequences are confirmed convincingly for model I by the simulation data shown in figure 3; similar results can be shown for model II. We conclude that for $p(\phi) = \frac{1}{2}\delta\big[\phi - \frac{1}{4}\pi\big] + \frac{1}{2}\delta\big[\phi + \frac{1}{4}\pi\big]$ the phase diagram for both models I and II is identical to that of homogeneously distributed pinning angles, i.e. figure 2.

6.3.2. $p(\phi) = \delta[\phi]$. For $p(\phi) = \delta[\phi]$ we no longer have rotation-invariant solutions, and analysis becomes more complicated. Given the assumption of reflection symmetry ($Q_{cs} = q_{cs} = 0$, whose stability we calculate below) we are left with three RS order parameters, $\{Q_{cc}, q_{cc}, q_{ss}\}$, and with the effective measures ((34) and (35)) which now become

$$M_{\text{I}}(\theta|x, y) = \mathrm{e}^{\beta h\cos(\theta)+(\beta K)^2[2Q_{cc}-1+q_{ss}-q_{cc}]\cos(2\theta)+2\beta K[\cos(\theta)x\sqrt{q_{cc}}+\sin(\theta)y\sqrt{q_{ss}}\,]} \tag{118}$$

$$M_{\text{II}}(\theta|x, y) = \mathrm{e}^{\beta h\cos(\theta)-(\beta K)^2[2Q_{cc}-1+q_{ss}-q_{cc}]\cos(2\theta)+2\beta K[\cos(\theta)x\sqrt{q_{ss}}+\sin(\theta)y\sqrt{q_{cc}}\,]}. \tag{119}$$

The remaining order parameters are to be solved from

$$Q_{cc} = \langle\langle\langle\langle\cos^2(\theta)\rangle_*\rangle\rangle\rangle_\phi \qquad q_{cc}\langle\langle\langle\langle\cos(\theta)\rangle^2_*\rangle\rangle\rangle_\phi \qquad q_{ss} = \langle\langle\langle\langle\sin(\theta)\rangle^2_*\rangle\rangle\rangle_\phi. \tag{120}$$

At the RS ground state we know that $q_{cc} + q_{ss} = 1$ and $Q_{cc} = \frac{1}{2}[1 + q_{cc} - q_{ss}]$. Due to $p(\phi) = \delta[\phi]$ the RS susceptibility (45) simplifies to

$$\chi_{\text{RS}} = \beta(1 - Q_{cc} - q_{ss}). \tag{121}$$

For strong pinning fields one finds $\lim_{h\to\infty} Q_{cc} = \lim_{h\to\infty} q_{cc} = 1$ and $\lim_{h\to\infty} q_{ss} = 0$. Hence $\lim_{h\to\infty}\chi_{\mathrm{RS}} = 0$. In the high-temperature regime an expansion in powers of β shows the solution of (120) to behave as

$$Q_{cc} = \tfrac{1}{2} + \mathcal{O}(\beta^2) \qquad q_{cc} = \tfrac{1}{4}\beta^2 h^2 + \mathcal{O}(\beta^3) \qquad q_{ss} = \mathcal{O}(\beta^3). \tag{122}$$

Hence $\chi_{\mathrm{RS}} = 1/2T + \mathcal{O}(T^{-3})$ as $T \to \infty$.

The expressions for the AT line(s) ((43) and (44)) and reflection symmetry transitions ((47) and (48)) cannot be simplified further, except in special limits and in the special case described below. For $h \to \infty$ one trivially extracts from these equations that both replica symmetry and reflection symmetry are stable for all finite temperatures; the same is true for high temperatures and arbitrary field strengths (as expected).

Yet, analytical progress can be made for model I. Here, in line with (122), our order parameter equations allow for solutions with $q_{ss} = 0$. Given the identification $q_{ss} = \lim_{N\to\infty} N^{-1}\sum_i \overline{\langle\sin(\theta_i)\rangle^2}$, such solutions imply that $\langle\sin(\theta_i)\rangle = 0$ for all i. This can be understood as a result of the action of the pinning fields, which for $p(\phi) = \delta(\phi)$ drive the phases towards $\phi_i = 0$. Insertion of $q_{ss} = 0$ as an ansatz into (118) gives $\langle\sin(\theta)\rangle_* = \langle\sin(\theta)\cos(\theta)\rangle_* = 0$, confirming $q_{ss} = 0$ self-consistently (this is not possible for model II). This results in considerable simplifications, such as $\gamma_{cs} = 0$ and $\gamma_{ss} = \langle\sin^2(\theta)\rangle_*$, and reduces the computational effort in the various numerical integrations. One now has only two order parameters, Q_{cc} and q_{cc}, to be solved from

$$Q_{cc} = \int Dx\,\langle\cos^2(\theta)\rangle_* \qquad q_{cc} = \int Dx\,\langle\cos(\theta)\rangle_*^2 \tag{123}$$

with

$$M_{\mathrm{I}}(\theta|x) = \mathrm{e}^{\beta\cos(\theta)[h+2Kx\sqrt{q_{cc}}]+(\beta K)^2[2Q_{cc}-1-q_{cc}]\cos(2\theta)}. \tag{124}$$

The ground state has $Q_{cc} = q_{cc} = 1$. The equation for the AT line of model I becomes

$$\left(\frac{T}{2K}\right)^2 = \max\left\{\int Dx[\langle\cos^2(\theta)\rangle_* - \langle\cos(\theta)\rangle_*^2]^2, \int Dx\,\langle\sin^2(\theta)\rangle_*^2\right\}. \tag{125}$$

Similarly we can for $q_{ss} = 0$ simplify the constituents $\{\lambda_i\}$ of (47) to

$$\lambda_1 = 2\int Dx\,\langle\sin^2(\theta)\cos^2(\theta)\rangle_* \tag{126}$$

$$\lambda_2 = 2\int Dx\,\langle\sin^2(\theta)\rangle_*[\langle\cos^2(\theta)\rangle_* - 2\langle\cos(\theta)\rangle_*^2] \tag{127}$$

$$\lambda_3 = 4\int Dx\,\langle\sin^2(\theta)\cos^2(\theta)\rangle_*\langle\cos(\theta)\rangle_*. \tag{128}$$

These are to be inserted into (47), whose zeros mark reflection symmetry breaking. We now also have a third type of transition: destabilization of $q_{ss} = 0$. The condition for this, $\partial^2\overline{f}_I[\cdots]/\partial q_{ss}^2 = 0$, with appendix A can be written as

$$(T/2K)^2 = \int Dx\,\langle\sin^2(\theta)\rangle_*^2. \tag{129}$$

Clearly, for $h \to 0$ (absent pinning fields) the $q_{ss} \neq 0$ bifurcation occurs at $T = K$, and coincides with a bifurcation of $q_{cc} \neq 0$ and with the AT line. Comparison with (125) shows that the critical temperature T_{c} defined by (129) obeys $T_{\mathrm{c}} \leqslant T_{\mathrm{AT}}$. Hence the AT instability does occur for $q_{ss} = 0$ and is thus given by (125). Numerical analysis reveals that along the line (125) one always has $\int Dx[\langle\cos^2(\theta)\rangle_* - \langle\cos(\theta)\rangle_*^2]^2 \leqslant \int Dx\,\langle\sin^2(\theta)\rangle_*^2$, and hence $T_{\mathrm{AT}} = T_{\mathrm{c}}$ (the two transition lines coincide, in agreement with [26, 27]).

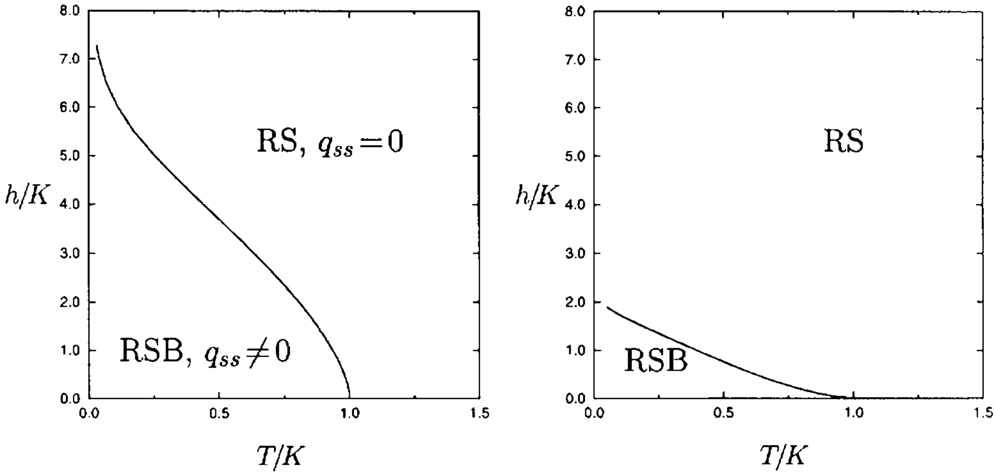

Figure 4. The phase diagrams for models I (left) and II (right), in the case of $p(\phi) = \delta[\phi]$, Solid line: the AT instability, signaling a second order transition from a replica symmetric state (RS) to a state without replica symmetry (RSB). For model I the AT line coincides with the line marking a continuous bifurcation of $q_{ss} \neq 0$ solutions of the RS order parameter equations.

Numerical solutions of (43) and (44) and (129) lead to the phase diagrams in figure 4. The $q_{ss} \neq 0$ transition for model I coincides with the AT line. The RS stabilizing effect of the pinning fields is apparently much greater for model II than for model I. Numerical evaluation of the conditions ((47) and (48)) marking the breaking of reflection symmetry shows for both models that this does not happen at any finite temperature.

The results of comparing the solutions of our RS equations (obtained numerically) with simulations are shown in figures 5 and 6. It is clear from these figures that, due to the breaking of global rotation symmetry by the external pinning fields, there is now a profound difference between the macroscopic behaviour of our models. Energetic conflicts are resolved differently. With the exception of the susceptibility, the agreement between theory and numerical experiment is good, taking into account finite size effects which lead as usual to a smoothing of the second order phase transition marking $q_{ss} \neq 0$ bifurcations. Comparison with the locations of the AT lines in figure 4 suggests again that the serious deviations in the susceptibility are due to replica symmetry breaking; $\chi_{\rm RS}$ cannot be expected to be correct in the RSB region.

6.3.3. $p(\phi) = \frac{1}{2}\delta\left[\phi - \frac{1}{2}\pi\right] + \frac{1}{2}\delta\left[\phi + \frac{1}{2}\pi\right]$. Our final example is a bimodal distribution which, due to $\langle\cos(2\theta)\rangle_\phi = -1 \neq 0$, again does not allow for rotation-invariant states. According to (4) (with $\eta = \lambda = 0$) this choice is related to the case $p(\phi) = \delta[\phi]$ by a simple gauge transformation, and our models must have identical free energies and phase diagrams. As a consistency test we will try to extract this property from the RS saddle-point equations. Given reflection symmetry, the two effective measures (34) and (35) now become

$$M_{\rm I}\left(\theta|x, y \pm \frac{\pi}{2}\right) = {\rm e}^{\pm\beta h\sin(\theta)+(\beta K)^2[2Q_{cc}-1+q_{ss}-q_{cc}]\cos(2\theta)+2\beta K[\cos(\theta)x\sqrt{q_{cc}}+\sin(\theta)y\sqrt{q_{ss}}]} \tag{130}$$

$$M_{\rm II}\left(\theta|x, y \pm \frac{\pi}{2}\right) = {\rm e}^{\pm\beta h\sin(\theta)-(\beta K)^2[2Q_{cc}-1+q_{ss}-q_{cc}]\cos(2\theta)+2\beta K[\cos(\theta)x\sqrt{q_{ss}}+\sin(\theta)y\sqrt{q_{cc}}]}. \tag{131}$$

We see that our models can indeed be mapped onto those obtained for $p(\phi) = \delta[\phi]$. We introduce the transformation $\theta = \pm\left(\frac{1}{2}\pi - \theta'\right)$ (permutation and reflection of axes), which,

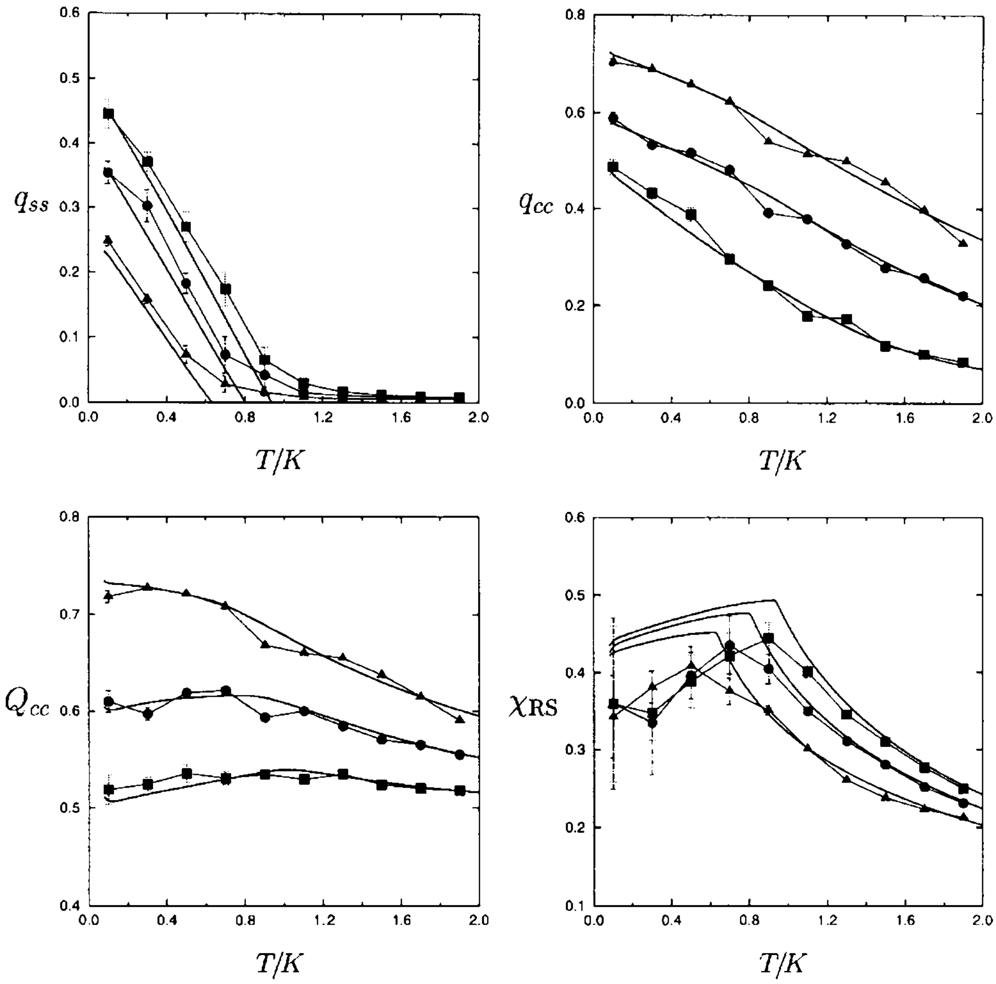

Figure 5. The theoretical predictions for the RS order parameters $\{q_{ss}, q_{cc}, Q_{cc}\}$ and the RS susceptibility, χ_{RS} as functions of temperature, for model I with $K = 1$, in the case of $p(\phi) = \delta[\phi]$. Thick solid lines: theoretical predictions. Markers: results of numerical simulations with $N = 800$. Different curves correspond to different values of the pinning field strength h, taken from $\{1, 2, 3\}$ (indicated by squares, circles and triangles, respectively).

via the order parameter equations (120), induces a corresponding transformation of the order parameters:

$$Q_{cc} = 1 - Q'_{cc} \qquad q_{cc} = q'_{ss} \qquad q_{ss} = q'_{cc}. \tag{132}$$

The effect on the measures (130) and (131) is that the latter can now be expressed in terms of the measures (118) and (119) describing the distribution $p(\phi) = \delta[\phi]$ studied previously:

$$M_{\mathrm{I}}\left(\theta|x, y \pm \frac{\pi}{2}\right) = M_{\mathrm{I}}(\theta'|\pm y, x) \qquad M_{\mathrm{II}}\left(\theta|x, y, \pm\frac{\pi}{2}\right) = M_{\mathrm{II}}(\theta'|\pm y, x).$$

Since operations such as $(x, y) \to (\pm y, x)$ have no physical consequences, and since all observables and transition lines are (within RS) constructed from the measures (130) and (131), we conclude that the macroscopic physics of the cases $p(\phi) = \delta[\phi]$ and $p(\phi) = \frac{1}{2}\delta\left[\phi - \frac{1}{2}\pi\right] + \frac{1}{2}\delta\left[\phi + \frac{1}{2}\pi\right]$ are indeed identical. The relations (132) map the order

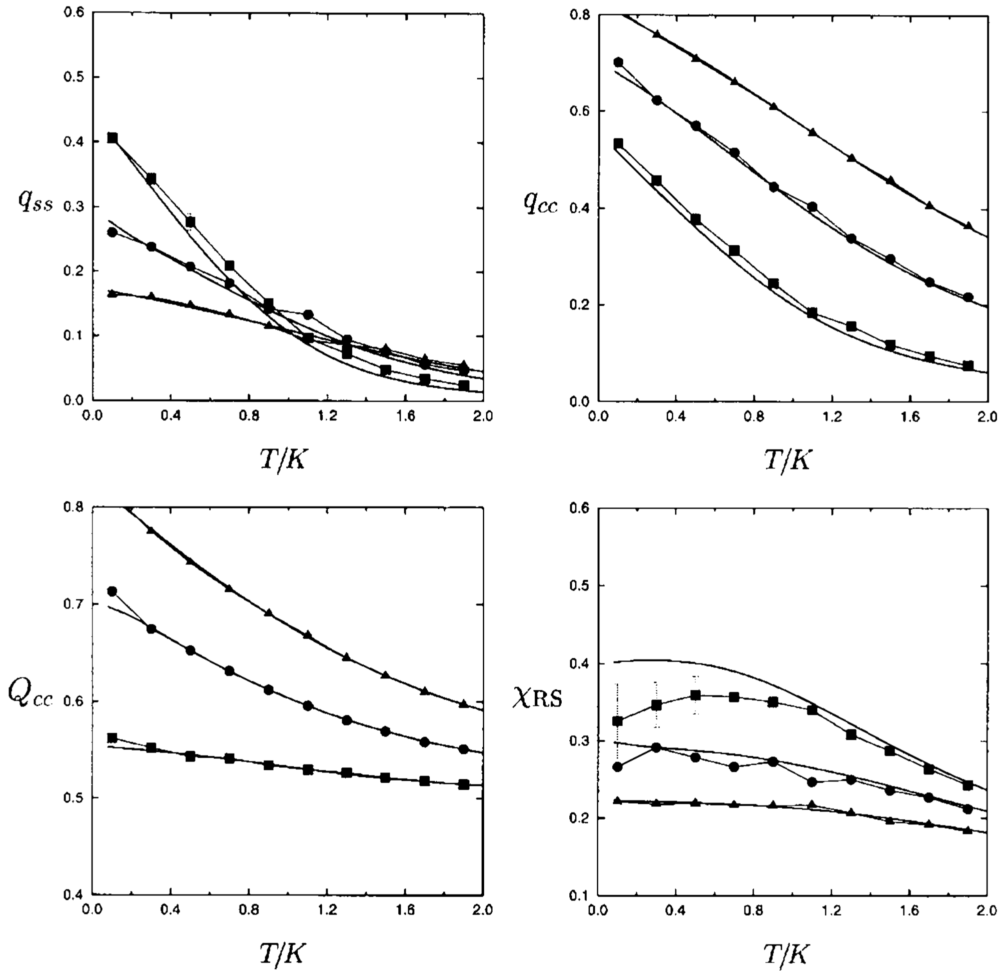

Figure 6. The theoretical predictions for the RS order parameters $\{q_{ss}, q_{cc}, Q_{cc}\}$ and the RS susceptibility $\chi_{\rm RS}$ as functions of temperature, for model II with $K = 1$, in the case of $p(\phi) = \delta[\phi]$. Thick solid lines: theoretical predictions. Markers: results of numerical simulations with $N = 800$. Different curves correspond to different values of the pinning field strength h, taken from $\{1, 2, 3\}$ (indicated by squares, circles and triangles, respectively).

parameters of the present bimodal distribution to those of the case $p(\phi) = \delta[\phi]$, and both cases have identical free energies and identical phase diagrams. In particular, the $q_{ss} = 0$ solution of model I for $p(\phi) = \delta[\phi]$ corresponds here to a solution with $q_{cc} = 0$ (again for model I only), and the RS susceptibility becomes $\chi_{\rm RS} = \beta(Q_{cc} - q_{cc})$. The equivalence of the $p(\phi) = \delta[\phi]$ and $p(\phi) = \frac{1}{2}\delta\big[\phi - \frac{1}{2}\pi\big] + \frac{1}{2}\delta\big[\phi + \frac{1}{2}\pi\big]$ models is also immediately obvious in numerical simulations (which we do not show for brevity).

7. Discussion

We studied models of frustrated coupled Kuramoto oscillators in the presence of random pinning fields. To simplify our problem we assumed that the natural frequencies of all oscillators are the same and that the pinning field distribution is reflection-symmetric, i.e.

$p(\phi) = p(-\phi)$ (a simple gauge transform allows one to map the present models to those with larger families of pinning field distributions). We calculated the disorder-averaged free energy using the replica method for two types of random pair interactions (model I and model II, the first with standard ferro- and anti-ferromagnetic interactions and the second with chiral interactions), which differ in the level of frustration. In terms of, e.g., oscillator triplets, model I is partially bond-frustrated, whereas model II is fully bond-frustrated. For small system sizes the two models have significantly different ground state properties. Our main interest here was to understand the difference(s) in physical behaviour between the two models in equilibrium, in the thermodynamic limit. At the mathematical level our two models are found to differ in the details of the effective (replicated) single-spin measure, but they have the same types of order parameters. Within the replica symmetric ansatz, one finds different expressions for the AT instabilities; more unexpectedly, the two models are also found to differ in the nature of the extremum of the free energy.

We inspected the effects of different types of symmetries which our models could inherit from the pinning field distribution. Global reflection symmetry leads to a simplified measure and a reduced number of order parameters. Reflection symmetry breaking transitions are found to be possible at most in the RSB regime. In the case of global rotation symmetry, only one order parameter remains (within RS), and the effective measure, the free energy, and the AT lines are identical for both models. We analysed our order parameter equations in detail for four specific choices for the pinning field distribution, and tested our predictions against computer simulations. Without pinning fields, our two models behave identically to a standard long-range XY model with nonchiral Gaussian interactions [32]. The same is found to be true for uniformly distributed pinning angles. The agreement between theory and simulation is quite satisfactory, modulo finite size effects, except for the susceptibility at low temperatures, where RSB effects in combination with equilibration difficulties play a major role. We then studied inhomogeneous pinning angle distributions, of the form $p(\phi) = \frac{1}{2}\delta[\phi - \alpha] + \frac{1}{2}\delta[\phi + \alpha]$. For the special value $\alpha = \frac{\pi}{4}$ one finds, remarkably, that both of our models are again always in a rotation-invariant state (in spite of the fact that the pinning field distribution lacks rotation invariance); this is confirmed by simulations. For $\alpha = 0$ the symmetry between our two models is broken. They now differ significantly in terms of the location of the AT line, which for model I in addition coincides with a further order parameter bifurcation, as in [26, 27]. The $\alpha = \frac{\pi}{2}$ case can be mapped onto the $\alpha = 0$ one, by a suitable gauge transformation. Again, there is a good agreement between theory and simulations.

As a next stage it would be interesting to explore the behaviour of these models for oscillators with random distributions of natural frequencies, as well as in the presence of more general chiral interactions. In both extensions/generalizations the standard equilibrium replica formalism can no longer be used; in the first case one has to rely on dynamical formalisms [5, 6], whereas in the second case one has to call upon renormalization tricks [20].

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Appendix A. Derivatives of the RS free energy

First and second order derivatives of $f[\cdots]$ in (21) with respect to RS order parameters are calculated by working out the following relations, where γ and γ' denote any two order parameters from the set $\{Q_{**}, q_{**}\}$ (the disorder variables $\{x, y, u, v\}$ generated by the differentiations are subsequently eliminated via integration by parts):

$$\frac{1}{\beta K^2}\frac{\partial f}{\partial \gamma} = \frac{\partial U}{\partial \gamma} - \frac{1}{\beta^2 K^2}\left\langle\!\!\left\langle\!\!\left\langle\!\!\left\langle \frac{\partial \log M(\cdots)}{\partial \gamma}\right\rangle_* \right\rangle\!\!\right\rangle\!\!\right\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f}{\partial \gamma \partial \gamma'} = \frac{\partial^2 U}{\partial \gamma \partial \gamma'} - \frac{1}{\beta^2 K^2}\left\langle\!\!\left\langle\!\!\left\langle\!\!\left\langle \frac{\partial^2 \log M(\cdots)}{\partial \gamma \partial \gamma'}\right\rangle_* + \left\langle \frac{\partial \log M(\cdots)}{\partial \gamma}\frac{\partial \log M(\cdots)}{\partial \gamma'}\right\rangle_* - \left\langle \frac{\partial \log M(\cdots)}{\partial \gamma}\right\rangle_* \left\langle \frac{\partial \log M(\cdots)}{\partial \gamma'}\right\rangle_* \right\rangle\!\!\right\rangle\!\!\right\rangle_\phi$$

Appendix A.1. First order derivatives

$$\frac{1}{\beta K^2}\frac{\partial f_{\rm I}}{\partial Q_{cc}} = 4\{Q_{cc} - \langle\langle\langle\langle \cos^2(\theta)\rangle_*\rangle\rangle\rangle_\phi\}$$
$$\frac{1}{\beta K^2}\frac{\partial f_{\rm I}}{\partial Q_{cs}} = 4\{Q_{cs} - \langle\langle\langle\langle \sin(\theta)\cos(\theta)\rangle_*\rangle\rangle\rangle_\phi\}$$
$$\frac{1}{\beta K^2}\frac{\partial f_{\rm I}}{\partial q_{cc}} = 2\{\langle\langle\langle\langle \cos(\theta)\rangle_*^2\rangle\rangle\rangle_\phi - q_{cc}\}$$
$$\frac{1}{\beta K^2}\frac{\partial f_{\rm I}}{\partial q_{ss}} = 2\{\langle\langle\langle\langle \sin(\theta)\rangle_*^2\rangle\rangle\rangle_\phi - q_{ss}\}$$
$$\frac{1}{\beta K^2}\frac{\partial f_{\rm I}}{\partial q_{cs}} = 4\{\langle\langle\langle\langle \cos(\theta)\rangle_*\langle\sin(\theta)\rangle_*\rangle\rangle\rangle_\phi - q_{cs}\}$$
$$\frac{1}{\beta K^2}\frac{\partial f_{\rm II}}{\partial Q_{cc}} = -4\{Q_{cc} - \langle\langle\langle\langle \cos^2(\theta)\rangle_*\rangle\rangle\rangle_\phi\}$$
$$\frac{1}{\beta K^2}\frac{\partial f_{\rm II}}{\partial Q_{cs}} = -4\{Q_{cs} - \langle\langle\langle\langle \sin(\theta)\cos(\theta)\rangle_*\rangle\rangle\rangle_\phi\}$$
$$\frac{1}{\beta K^2}\frac{\partial f_{\rm II}}{\partial q_{cc}} = 2\{\langle\langle\langle\langle \sin(\theta)\rangle_*^2\rangle\rangle\rangle_\phi - q_{ss}\}$$
$$\frac{1}{\beta K^2}\frac{\partial f_{\rm II}}{\partial q_{ss}} = 2\{\langle\langle\langle\langle \cos(\theta)\rangle_*^2\rangle\rangle\rangle_\phi - q_{cc}\}$$
$$\frac{1}{\beta K^2}\frac{\partial f_{\rm II}}{\partial q_{cs}} = -4\{\langle\langle\langle\langle \cos(\theta)\rangle_*\langle\sin(\theta)\rangle_*\rangle\rangle\rangle_\phi - q_{cs}\}$$

Appendix A.2. Second order derivatives for model I

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm I}}{\partial Q_{cc}^2} = 4 - 4(\beta K)^2\langle\langle\langle\langle \cos^2(2\theta)\rangle_* - \langle\cos(2\theta)\rangle_*^2\rangle\rangle\rangle_\phi$$
$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm I}}{\partial Q_{cs}^2} = 4 - 4(\beta K)^2\langle\langle\langle\langle \sin^2(2\theta)\rangle_* - \langle\sin(2\theta)\rangle_*^2\rangle\rangle\rangle_\phi$$
$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm I}}{\partial Q_{cc}\partial Q_{cs}} = -4(\beta K)^2\langle\langle\langle\langle \sin(2\theta)\cos(2\theta)\rangle_* - \langle\sin(2\theta)\rangle_*\langle\cos(2\theta)\rangle_*\rangle\rangle\rangle_\phi$$
$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm I}}{\partial Q_{cc}\partial q_{cc}} = 8(\beta K)^2\langle\langle\langle\langle \cos(2\theta)\cos(\theta)\rangle_*\langle\cos(\theta)\rangle_* - \langle\cos(2\theta)\rangle_*\langle\cos(\theta)\rangle_*^2\rangle\rangle\rangle_\phi$$
$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm I}}{\partial Q_{cc}\partial q_{ss}} = 8(\beta K)^2\langle\langle\langle\langle \cos(2\theta)\sin(\theta)\rangle_*\langle\sin(\theta)\rangle_* - \langle\cos(2\theta)\rangle_*\langle\sin(\theta)\rangle_*^2\rangle\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm I}}{\partial Q_{cc}\partial q_{cs}} = 8(\beta K)^2\langle\langle\!\langle\langle\cos(2\theta)\cos(\theta)\rangle_*\langle\sin(\theta)\rangle_* + \langle\cos(2\theta)\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_* - 2\langle\cos(2\theta)\rangle_*\langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm I}}{\partial Q_{cs}\partial q_{cc}} = 8(\beta K)^2\langle\langle\!\langle\langle\sin(2\theta)\cos(\theta)\rangle_*\langle\cos(\theta)\rangle_* - \langle\sin(2\theta)\rangle_*\langle\cos(\theta)\rangle_*^2\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm I}}{\partial Q_{cs}\partial q_{ss}} = 8(\beta K)^2\langle\langle\!\langle\langle\sin(2\theta)\sin(\theta)\rangle_*\langle\sin(\theta)\rangle_* - \langle\sin(2\theta)\rangle_*\langle\sin(\theta)\rangle_*^2\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm I}}{\partial Q_{cs}\partial q_{cs}} = 8(\beta K)^2\langle\langle\!\langle\langle\sin(2\theta)\cos(\theta)\rangle_*\langle\sin(\theta)\rangle_* + \langle\sin(2\theta)\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_* - 2\langle\sin(2\theta)\rangle_*\langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm I}}{\partial q_{cc}^2} = -2 + 8(\beta K)^2\langle\langle\!\langle[\langle\cos^2(\theta)\rangle_* - \langle\cos(\theta)\rangle_*^2][\langle\cos^2(\theta)\rangle_* - 3\langle\cos(\theta)\rangle_*^2]\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm I}}{\partial q_{ss}^2} = -2 + 8(\beta K)^2\langle\langle\!\langle[\langle\sin^2(\theta)\rangle_* - \langle\sin(\theta)\rangle_*^2][\langle\sin^2(\theta)\rangle_* - 3\langle\sin(\theta)\rangle_*^2]\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm I}}{\partial q_{cs}^2} = -4 + 8(\beta K)^2\langle\langle\!\langle[\langle\sin^2(\theta)\rangle_* - \langle\sin(\theta)\rangle_*^2][\langle\cos^2(\theta)\rangle_* - 3\langle\cos(\theta)\rangle_*^2] + [\langle\cos^2(\theta)\rangle_* - \langle\cos(\theta)\rangle_*^2][\langle\sin^2(\theta)\rangle_* - 3\langle\sin(\theta)\rangle_*^2] + 2[\langle\sin(\theta)\cos(\theta)\rangle_* - \langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*][\langle\sin(\theta)\cos(\theta)\rangle_* - 3\langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*]\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm I}}{\partial q_{cc}\partial q_{ss}} = 8(\beta K)^2\langle\langle\!\langle[\langle\sin(\theta)\cos(\theta)\rangle_* - \langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*] \times [\langle\sin(\theta)\cos(\theta)\rangle_* - 3\langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*]\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm I}}{\partial q_{cc}\partial q_{cs}} = 8(\beta K)^2\langle\langle\!\langle[\langle\sin(\theta)\cos(\theta)\rangle_* - \langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*][\langle\cos^2(\theta)\rangle_* - 3\langle\cos(\theta)\rangle_*^2] + [\langle\sin(\theta)\cos(\theta)\rangle_* - 3\langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*][\langle\cos^2(\theta)\rangle_* - \langle\cos(\theta)\rangle_*^2]\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm I}}{\partial q_{ss}\partial q_{cs}} = 8(\beta K)^2\langle\langle\!\langle[\langle\sin(\theta)\cos(\theta)\rangle_* - \langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*][\langle\sin^2(\theta)\rangle_* - 3\langle\sin(\theta)\rangle_*^2] + [\langle\sin(\theta)\cos(\theta)\rangle_* - 3\langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*][\langle\sin^2(\theta)\rangle_* - \langle\sin(\theta)\rangle_*^2]\rangle\!\rangle\rangle_\phi.$$

Appendix A.3. Second order derivatives for model II

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm II}}{\partial Q_{cc}^2} = -4 - 4(\beta K)^2\langle\langle\!\langle\langle\cos^2(2\theta)\rangle_* - \langle\cos(2\theta)\rangle_*^2\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm II}}{\partial Q_{cs}^2} = -4 - 4(\beta K)^2\langle\langle\!\langle\langle\sin^2(2\theta)\rangle_* - \langle\sin(2\theta)\rangle_*^2\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm II}}{\partial Q_{cc}\partial Q_{cs}} = -4(\beta K)^2\langle\langle\!\langle\langle\sin(2\theta)\cos(2\theta)\rangle_* - \langle\sin(2\theta)\rangle_*\langle\cos(2\theta)\rangle_*\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm II}}{\partial Q_{cc}\partial q_{cc}} = -8(\beta K)^2\langle\langle\!\langle\langle\cos(2\theta)\sin(\theta)\rangle_*\langle\sin(\theta)\rangle_* - \langle\cos(2\theta)\rangle_*\langle\sin(\theta)\rangle_*^2\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm II}}{\partial Q_{cc}\partial q_{ss}} = -8(\beta K)^2\langle\langle\!\langle\langle\cos(2\theta)\cos(\theta)\rangle_*\langle\cos(\theta)\rangle_* - \langle\cos(2\theta)\rangle_*\langle\cos(\theta)\rangle_*^2\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm II}}{\partial Q_{cc}\partial q_{cs}} = 8(\beta K)^2\langle\langle\!\langle\langle\cos(2\theta)\cos(\theta)\rangle_*\langle\sin(\theta)\rangle_* + \langle\cos(2\theta)\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_* - 2\langle\cos(2\theta)\rangle_*\langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm II}}{\partial Q_{cs}\partial q_{cc}} = -8(\beta K)^2\langle\langle\!\langle\langle\sin(2\theta)\sin(\theta)\rangle_*\langle\sin(\theta)\rangle_* - \langle\sin(2\theta)\rangle_*\langle\sin(\theta)\rangle_*^2\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm II}}{\partial Q_{cs}\partial q_{ss}} = -8(\beta K)^2\langle\langle\!\langle\langle\sin(2\theta)\cos(\theta)\rangle_*\langle\cos(\theta)\rangle_* - \langle\sin(2\theta)\rangle_*\langle\cos(\theta)\rangle_*^2\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm II}}{\partial Q_{cs}\partial q_{cs}} = 8(\beta K)^2\langle\langle\!\langle\langle\sin(2\theta)\cos(\theta)\rangle_*\langle\sin(\theta)\rangle_* + \langle\sin(2\theta)\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_* - 2\langle\sin(2\theta)\rangle_*\langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm II}}{\partial q_{cc}^2} = 8(\beta K)^2\langle\langle\!\langle[\langle\sin^2(\theta)\rangle_* - \langle\sin(\theta)\rangle_*^2][\langle\sin^2(\theta)\rangle_* - 3\langle\sin(\theta)\rangle_*^2]\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm II}}{\partial q_{ss}^2} = 8(\beta K)^2\langle\langle\!\langle[\langle\cos^2(\theta)\rangle_* - \langle\cos(\theta)\rangle_*^2][\langle\cos^2(\theta)\rangle_* - 3\langle\cos(\theta)\rangle_*^2]\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm II}}{\partial q_{cs}^2} = 4 + 8(\beta K)^2\langle\langle\!\langle[\langle\sin^2(\theta)\rangle_* - \langle\sin(\theta)\rangle_*^2][\langle\cos^2(\theta)\rangle_* - 3\langle\cos(\theta)\rangle_*^2] + [\langle\cos^2(\theta)\rangle_* - \langle\cos(\theta)\rangle_*^2][\langle\sin^2(\theta)\rangle_* - 3\langle\sin(\theta)\rangle_*^2] + 2[\langle\sin(\theta)\cos(\theta)\rangle_* - \langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*][\langle\sin(\theta)\cos(\theta)\rangle_* - 3\langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*]\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm II}}{\partial q_{cc}\partial q_{ss}} = -2 + 8(\beta K)^2\langle\langle\!\langle[\langle\sin(\theta)\cos(\theta)\rangle_* - \langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*] \times [\langle\sin(\theta)\cos(\theta)\rangle_* - 3\langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*]\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm II}}{\partial q_{cc}\partial q_{cs}} = -8(\beta K)^2\langle\langle\!\langle[\langle\sin(\theta)\cos(\theta)\rangle_* - \langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*][\langle\sin^2(\theta)\rangle_* - 3\langle\sin(\theta)\rangle_*^2] + [\langle\sin(\theta)\cos(\theta)\rangle_* - 3\langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*][\langle\sin^2(\theta)\rangle_* - \langle\sin(\theta)\rangle_*^2]\rangle\!\rangle\rangle_\phi$$

$$\frac{1}{\beta K^2}\frac{\partial^2 f_{\rm II}}{\partial q_{ss}\partial q_{cs}} = -8(\beta K)^2\langle\langle\!\langle[\langle\sin(\theta)\cos(\theta)\rangle_* - \langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*][\langle\cos^2(\theta)\rangle_* - 3\langle\cos(\theta)\rangle_*^2] + [\langle\sin(\theta)\cos(\theta)\rangle_* - 3\langle\sin(\theta)\rangle_*\langle\cos(\theta)\rangle_*][\langle\cos^2(\theta)\rangle_* - \langle\cos(\theta)\rangle_*^2]\rangle\!\rangle\rangle_\phi.$$

Appendix B. Derivation of the AT instability

Appendix B.1. Calculation of the RS Hessian for replicon fluctuations

We calculate the Hessian of the disorder-averaged free energy per oscillator by expanding (7) in powers of the fluctuations $\{\delta\boldsymbol{q}^{**}, \delta\hat{\boldsymbol{q}}^{**}\}$ around the RS saddle point. We use pairs of Roman indices (ab) and (de) to label the four combinations $\{cc, ss, cs, sc\}$, and abbreviate the corresponding functions in the obvious way as $a(\theta), b(\theta), c(\theta), d(\theta) \in \{\cos(\theta), \sin(\theta)\}$. We put $\hat{q}^{**}_{\alpha\beta} = 2{\rm i}(\beta K)^2 k^{**}_{\alpha\beta}$ and $\delta\overline{f}[\cdots] = \overline{f}[\cdots] - \overline{f}_{\rm RS}[\cdots]$, and obtain:

$$-\frac{n}{\beta K^2}\delta\overline{f}[\cdots] = 2\sin^2(A^*)\sum_{\alpha\beta}\left[\delta q^{ss}_{\alpha\beta}\delta q^{cc}_{\alpha\beta} - \delta q^{sc}_{\alpha\beta}\delta q^{cs}_{\alpha\beta}\right] + \cos^2(A^*)\sum_{ab}\sum_{\alpha\beta}\left[\delta q^{ab}_{\alpha\beta}\right]^2 - 2\sum_{ab}\sum_{\alpha\beta}\delta k^{ab}_{\alpha\beta}\delta q^{ab}_{\alpha\beta} + \frac{1}{(\beta K)^2}\left\langle \log\left\{\frac{\int {\rm d}\boldsymbol{\theta}\, M(\boldsymbol{\theta}|\{\boldsymbol{k}^{**}+\delta\boldsymbol{k}^{**}\})}{\int {\rm d}\boldsymbol{\theta}\, M(\boldsymbol{\theta}|\{\boldsymbol{k}^{**}\})}\right\}\right\rangle_\phi + \cdots \tag{B.1}$$

with

$$M(\boldsymbol{\theta}|\{\boldsymbol{k}^{**}\}) = \mathrm{e}^{\beta h\sum_\alpha \cos(\theta^\alpha-\phi)+2(\beta K)^2\sum_{ab}\sum_{\alpha\beta}\boldsymbol{k}^{ab}_{\alpha\beta}a(\theta_\alpha)b(\theta_\beta)}.$$

Working out the fraction in the last term of (B.1) gives

$$\frac{\int \mathrm{d}\boldsymbol{\theta}\, M(\boldsymbol{\theta}|\{\boldsymbol{k}^{**}+\delta\boldsymbol{k}^{**}\})}{\int \mathrm{d}\boldsymbol{\theta}\, M(\boldsymbol{\theta}|\{\boldsymbol{k}^{**}\})} = 1+2(\beta K)^2\sum_{\alpha\beta}\sum_{ab}\delta k^{ab}_{\alpha\beta}\frac{\int \mathrm{d}\boldsymbol{\theta}\, M(\boldsymbol{\theta}|\{\boldsymbol{k}^{**}\})a(\theta_\alpha)b(\theta_\beta)}{\int \mathrm{d}\boldsymbol{\theta}\, M(\boldsymbol{\theta}|\{\boldsymbol{k}^{**}\})}$$
$$+2(\beta K)^4\sum_{\alpha\beta\gamma\delta}\sum_{ab,de}\delta k^{ab}_{\alpha\beta}\delta k^{de}_{\gamma\delta}\frac{\int \mathrm{d}\boldsymbol{\theta}\, M(\boldsymbol{\theta}|\{\boldsymbol{k}^{**}\})a(\theta_\alpha)b(\theta_\beta)d(\theta_\gamma)e(\theta_\delta)}{\int \mathrm{d}\boldsymbol{\theta}\, M(\boldsymbol{\theta}|\{\boldsymbol{k}^{**}\})}+\cdots.$$

Upon expanding $\log(x) = x - \frac{1}{2}x^2 + \cdots$, and upon introducing the shorthand $M(\boldsymbol{\theta}|\mathrm{RS}) = M(\boldsymbol{\theta}|\{\boldsymbol{k}^{**}_{\mathrm{RS}}\})$, we arrive at

$$-\frac{n}{\beta K^2}\delta\overline{f}[\cdots] = 2\sin^2(A^*)\sum_{\alpha\beta}\left[\delta q^{ss}_{\alpha\beta}\delta q^{cc}_{\alpha\beta} - \delta q^{sc}_{\alpha\beta}\delta q^{cs}_{\alpha\beta}\right] + \cos^2(A^*)\sum_{ab}\sum_{\alpha\beta}\left[\delta q^{ab}_{\alpha\beta}\right]^2$$
$$-2\sum_{ab}\sum_{\alpha\beta}\delta k^{ab}_{\alpha\beta}\delta q^{ab}_{\alpha\beta} + 2(\beta K)^2\sum_{\alpha\beta\gamma\delta}\sum_{ab,de}\delta k^{ab}_{\alpha\beta}\delta k^{de}_{\gamma\delta}$$
$$\times\left\langle\frac{\int \mathrm{d}\boldsymbol{\theta}\, M(\boldsymbol{\theta}|\mathrm{RS})a(\theta_\alpha)b(\theta_\beta)d(\theta_\gamma)e(\theta_\delta)}{\int \mathrm{d}\boldsymbol{\theta}\, M(\boldsymbol{\theta}|\mathrm{RS})}\right.$$
$$\left. - \frac{\int \mathrm{d}\boldsymbol{\theta}\, M(\boldsymbol{\theta}|\mathrm{RS})a(\theta_\alpha)b(\theta_\beta)}{\int \mathrm{d}\boldsymbol{\theta}\, M(\boldsymbol{\theta}|\mathrm{RS})}\frac{\int \mathrm{d}\boldsymbol{\theta}\, M(\boldsymbol{\theta}|\mathrm{RS})d(\theta_\gamma)e(\theta_\delta)}{\int \mathrm{d}\boldsymbol{\theta}\, M(\boldsymbol{\theta}|\mathrm{RS})}\right\rangle_\phi + \cdots.$$

We work out the last of the above quadratic terms for fluctuations around the RS solution, for small n. We abbreviate $Dxyuv = Dx\,Dy\,Du\,Dv$, and denote by $M(\theta)$ either the measure (11) (for model I) or (12) (for model II):

$$\langle\cdots\rangle_\phi = \left\langle\frac{\int Dxyuv\left[\prod_\lambda\int \mathrm{d}\theta\, M(\theta_\lambda)\right]a(\theta_\alpha)b(\theta_\beta)c(\theta_\gamma)d(\theta_\delta)}{\int Dxyuv\left[\int \mathrm{d}\theta\, M(\theta)\right]^n}\right\rangle_\phi$$
$$-\left\langle\frac{\int Dxyuv\left[\prod_\lambda\int \mathrm{d}\theta_\lambda\, M(\theta_\lambda)\right]a(\theta_\alpha)b(\theta_\beta)}{\int Dxyuv\left[\int \mathrm{d}\theta\, M(\theta)\right]^n}\frac{\int Dxyuv\left[\prod_\lambda\int \mathrm{d}\theta_\lambda\, M(\theta_\lambda)\right]c(\theta_\gamma)d(\theta_\delta)}{\int Dxyuv\left[\int \mathrm{d}\theta\, M(\theta)\right]^n}\right\rangle_\phi.$$

Restricting ourselves to replicon fluctuations, i.e. $\delta q^{**}_{\alpha\alpha} = \delta k^{**}_{\alpha\alpha} = 0$, $\sum_\alpha \delta k_{\alpha\beta} = \sum_\alpha \delta q_{\alpha\beta} = 0$ and $\sum_\beta \delta k_{\alpha\beta} = \sum_\beta \delta q_{\alpha\beta} = 0$, we can proceed by inserting a string of Kronecker symbols (and complementary symbols $\overline{\delta}_{\alpha\beta} = 1 - \delta_{\alpha\beta}$) to do the bookkeeping of possibly identical combinations of replica indices:

$$\delta_{\alpha\delta}\delta_{\beta\gamma} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\overline{\delta}_{\beta\gamma} + \delta_{\alpha\gamma}\overline{\delta}_{\beta\delta} + \overline{\delta}_{\alpha\gamma}\delta_{\beta\delta} + \overline{\delta}_{\alpha\delta}\delta_{\beta\gamma} + \overline{\delta}_{\alpha\delta}\overline{\delta}_{\alpha\gamma}\overline{\delta}_{\beta\delta}\overline{\delta}_{\beta\gamma}$$

giving, for $n \to 0$, and upon using again the replicon properties:

$$\langle\cdots\rangle_\phi = \delta_{\alpha\delta}\delta_{\beta\gamma}\langle\langle\!\langle[\langle a(\theta)e(\theta)\rangle_* - \langle a(\theta)\rangle_*\langle e(\theta)\rangle_*][\langle b(\theta)d(\theta)\rangle_*$$
$$-\langle b(\theta)\rangle_*\langle d(\theta)\rangle_*]\rangle\!\rangle\rangle_\phi + \delta_{\beta\delta}\delta_{\beta\gamma}\langle\langle\!\langle[\langle b(\theta)e(\theta)\rangle_*$$
$$-\langle b(\theta)\rangle_*\langle e(\theta)\rangle_*][\langle a(\theta)d(\theta)\rangle_* - \langle a(\theta)\rangle_*\langle d(\theta)\rangle_*]\rangle\!\rangle\rangle_\phi.$$

The remaining terms give identical contributions to the fluctuations around the RS free energy, after the summation over $(\gamma, \delta; de)$, and the result can be written as

$$-\frac{n}{\beta K^2}\delta\overline{f}[\cdots] = \sum_{\alpha\neq\beta}\sum_{ab,de}\left\{\delta q^{ab}_{\alpha\beta}\delta q^{de}_{\alpha\beta}\left[\sin^2(A^*)C_{ab,de} + \cos^2(A^*)\delta_{ab,de}\right]\right.$$
$$\left. - 2\delta q_{\alpha\beta}\delta k^{de}_{\alpha\beta}\,\delta_{ab,de} + 4(\beta K)^2\delta k^{ab}_{\alpha\beta}\,\delta k^{de}_{\alpha\beta}E_{ab,de}\right\} + \cdots \tag{B.2}$$

with the two 4×4 matrices (note: $ab, de \in \{cc, ss, cs, sc\}$)

$$C_{ab,de} = \delta_{ab,ss}\delta_{de,cc} + \delta_{ab,cc}\delta_{de,ss} - \delta_{ab,cs}\delta_{de,sc} - \delta_{ab,sc}\delta_{de,cs} \tag{B.3}$$

$$E_{ab,de} = \langle\langle\!\langle[\langle b(\theta)e(\theta)\rangle_* - \langle b(\theta)\rangle_*\langle e(\theta)\rangle_*][\langle a(\theta)d(\theta)\rangle_* - \langle a(\theta)\rangle_*\langle d(\theta)\rangle_*]\rangle\!\rangle\rangle_\phi. \tag{B.4}$$

The replica indices have become irrelevant labels, and for $n \to 0$ and within the subspace of replicon fluctuations, the spectrum of the Hessian reduces to that of the following 8×8 matrix:

$$\mathcal{H} = \begin{pmatrix} \cos^2(A^*)\boldsymbol{I} + \sin^2 A^* \boldsymbol{C} & -\boldsymbol{I} \\ -\boldsymbol{I} & 4(\beta K)^2 \boldsymbol{E} \end{pmatrix} \tag{B.5}$$

(apart from an overall multiplicative constant), with the building blocks $\boldsymbol{C} = \{C_{ab,de}\}$ and $\boldsymbol{E} = \{E_{ab,de}\}$ given in (B.3) and (B.4) and with the 4×4 unit matrix $\boldsymbol{I}_{ab,de} = \delta_{ab,de}$.

Appendix B.2. Replicon instabilities

Requiring (B.5) to have a zero eigenvalue (replicon instability) leads us to the condition

$$\exists \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \neq \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix} : \begin{cases} [\cos^2(A^*)\boldsymbol{I} + \sin^2(A^*)\boldsymbol{C}]\boldsymbol{x} = \boldsymbol{y} \\ \boldsymbol{x} = 4(\beta K)^2 \boldsymbol{E}\boldsymbol{y}. \end{cases}.$$

Equivalently, $\det\{4(\beta K)^2[\cos^2(A^*)\boldsymbol{I} + \sin^2(A^*)\boldsymbol{C}]\boldsymbol{E} - \boldsymbol{I}\} = 0$. For our two models, I ($A^* = 0$) and II $\left(A^* = \frac{1}{2}\pi\right)$, this translates into, using $\boldsymbol{C}^2 = \boldsymbol{I}$,

$$\text{model I:} \ \det[\boldsymbol{E} - (T/2K)^2\boldsymbol{I}] = 0 \tag{B.6}$$

$$\text{model II:} \ \det[\boldsymbol{E} - (T/2K)^2\boldsymbol{C}] = 0. \tag{B.7}$$

In the representation where the entries of our matrices are ordered as $\{cc, ss, cs, sc\}$, and with the assistance of (19, 20), we find our matrices $\boldsymbol{C}$ and $\boldsymbol{E}$ acquire the form

$$\boldsymbol{C} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \boldsymbol{E} = \left\langle\!\!\left\langle\!\!\left\langle \begin{pmatrix} \gamma_{cc}^2 & \gamma_{cs}^2 & \gamma_{cc}\gamma_{cs} & \gamma_{cc}\gamma_{cs} \\ \gamma_{cs}^2 & \gamma_{ss}^2 & \gamma_{ss}\gamma_{cs} & \gamma_{ss}\gamma_{cs} \\ \gamma_{cc}\gamma_{cs} & \gamma_{ss}\gamma_{cs} & \gamma_{cc}\gamma_{ss} & \gamma_{cs}^2 \\ \gamma_{cc}\gamma_{cs} & \gamma_{ss}\gamma_{cs} & \gamma_{cs}^2 & \gamma_{cc}\gamma_{ss} \end{pmatrix} \right\rangle\!\!\right\rangle\!\!\right\rangle_\phi. \tag{B.8}$$

One observes that $|0\rangle = \frac{1}{2}\sqrt{2}(0, 0, \mathbf{1}, -1)$ is an eigenvector of both $\boldsymbol{C}$ (with eigenvalue 1) and $\boldsymbol{E}$ $\left(\text{with eigenvalue } \left\langle \langle\!\langle \gamma_{cc}\gamma_{ss} - \gamma_{cs}^2 \rangle\!\rangle \right\rangle_\phi\right)$, and therefore also of the matrices in (B.6) and (B.7). This leads us to the first replicon instability condition, for both models:

$$(T/2K)^2 = \left\langle \langle\!\langle \gamma_{cc}\gamma_{ss} - \gamma_{cs}^2 \rangle\!\rangle \right\rangle_\phi. \tag{B.9}$$

The remaining eigenvectors of the matrices in (B.6) and (B.7), namely $\boldsymbol{E} - (T/2K)^2\boldsymbol{I}$ (model I) and $\boldsymbol{E} - (T/2K)^2\boldsymbol{C}$ (model II), must be orthogonal to $|0\rangle$, and thus in the space spanned by the following orthogonal eigenvectors of $\boldsymbol{C}$ (with eigenvalues $\{1, -1, -1\}$):

$$|1\rangle = \tfrac{1}{2}\sqrt{2}(1, 1, 0, 0) \qquad |2\rangle = \tfrac{1}{2}\sqrt{2}(1, -1, 0, 0) \qquad |3\rangle = \tfrac{1}{2}\sqrt{2}(0, 0, 1, 1).$$

On the basis $\{|1\rangle, |2\rangle, |3\rangle\}$ the matrices (B.8) reduce to the following 3×3 ones

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{B.10}$$

$$
\boldsymbol{E} = \begin{pmatrix} \left\langle\left\langle\!\left\langle \frac{1}{2}\left(\gamma_{cc}^2+\gamma_{ss}^2\right)+\gamma_{cs}^2\right\rangle\!\right\rangle\right\rangle_\phi & \left\langle\left\langle\!\left\langle \frac{1}{2}\left(\gamma_{cc}^2-\gamma_{ss}^2\right)\right\rangle\!\right\rangle\right\rangle_\phi & \langle\langle\!\langle \gamma_{cs}(\gamma_{cc}+\gamma_{ss})\rangle\!\rangle\rangle_\phi \\ \left\langle\left\langle\!\left\langle \frac{1}{2}\left(\gamma_{cc}^2-\gamma_{ss}^2\right)\right\rangle\!\right\rangle\right\rangle_\phi & \left\langle\left\langle\!\left\langle \frac{1}{2}\left(\gamma_{cc}^2+\gamma_{ss}^2\right)-\gamma_{cs}^2\right\rangle\!\right\rangle\right\rangle_\phi & \langle\langle\!\langle \gamma_{cs}(\gamma_{cc}-\gamma_{ss})\rangle\!\rangle\rangle_\phi \\ \langle\langle\!\langle \gamma_{cs}(\gamma_{cc}+\gamma_{ss})\rangle\!\rangle\rangle_\phi & \langle\langle\!\langle \gamma_{cs}(\gamma_{cc}-\gamma_{ss})\rangle\!\rangle\rangle_\phi & \left\langle\left\langle\!\left\langle \gamma_{cc}\gamma_{ss}+\gamma_{cs}^2\right\rangle\!\right\rangle\right\rangle_\phi \end{pmatrix}. \tag{B.11}
$$

The remaining replicon instabilities thus follow upon inserting (B.10) and (B.11) into the conditions (B.6) and (B.7). The physical RSB transition associated with the combined replicon instabilities is the one occurring at the highest temperature.

References

[1] Winfree A T 1980 *The Geometry of Biological Time* (New York: Springer)
[2] Kuramoto Y 1984 *Chemical Oscillations, Waves and Turbulence* (Berlin: Springer)
[3] Kuramoto Y 1984 *Prog. Theor. Phys. Suppl.* **79** 223
[4] Strogatz S H and Mirollo R E 1991 *J. Stat. Phys.* **63** 613
[5] Bonilla L L, Neu J C and Spigler R 1992 *J. Stat. Phys.* **67** 313
[6] Bonilla L L, Perez Vicente C J and Spigler R 1998 *Physica* D **113** 79
[7] Acebron J A, Bonilla L L, De Leo S and Spigler R 1998 *Phys. Rev.* E **57** 5287
[8] Daido H 1988 *Phys. Rev. Lett.* **61** 231
[9] Strogatz S H and Mirollo R E 1988 *J. Phys. A: Math. Gen.* **21** L699
[10] Bonilla L L, Perez Vicente C J and Rubi J M 1993 *J. Stat. Phys.* **70** 921
[11] Arenas A and Perez Vicente C J 1994 *Europhys. Lett.* **26** 79
[12] Sakaguchi H 1988 *Prog. Theor. Phys.* **79** 39
[13] Reimann P, Van den Broeck C and Kawai R 1999 *Phys. Rev.* E **60** 6402
[14] Schuster H G and Wagner P 1990 *Biol. Cybern.* **64** 77
Schuster H G and Wagner P 1990 *Biol. Cybern.* **64** 83
[15] Abbott L F 1990 *J. Phys. A: Math. Gen.* **23** 3835
[16] Wiesenfeld K, Colet P and Strogatz S H 1996 *Phys. Rev. Lett.* **76** 404
Wiesenfeld K, Colet P and Strogatz S H 1998 *Phys. Rev.* E **57** 1563
[17] Granato E, Kosterlitz J M and Nightingale M P 1996 *Physica* B **222** 266
[18] Majhofer A 1996 *Physica* B **222** 273
[19] Sakaguchi H and Kuramoto Y 1986 *Prog. Theor. Phys.* **76** 576
[20] Rae H C, Coolen A C C and Sollich P 2002 in preparation
[21] Daido H 1992 *Phys. Rev. Lett.* **68** 1073
Daido H 2000 *Phys. Rev.* E **61** 2145
[22] Stiller J C and Radons G 1998 *Phys. Rev.* E **58** 1789
[23] Choi M Y and Doniach S 1985 *Phys. Rev.* B **31** 4516
[24] Arenas A and Perez Vicente C J 1994 *Phys. Rev.* E **50** 949
[25] Elderfield D J and Sherrington D 1982 *J. Phys. A: Math. Gen.* **15** L513
[26] Cragg D M, Sherrington D and Gabay M 1982 *Phys. Rev. Lett.* **49** 158
[27] Elderfield D and Sherrington D 1983 *J. Phys. C: Solid State Phys.* **16** 4865
[28] Park K, Rhee S W and Choi M Y 1998 *Phys. Rev.* E **57** 5030
[29] De Almeida J R L and Thouless D J 1978 *J. Phys. A: Math. Gen.* **11** 983
[30] Mézard M, Parisi G and Virasoro M A 1987 *Spin-Glass Theory and Beyond* (Singapore: World Scientific)
[31] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (New York: Dover)
[32] Kirkpatrick S and Sherrington D 1978 *Phys. Rev.* B **17** 4384